Efficient Social Welfare Function and Optimal Income Distribution

Jiandong Ju[1]

Department of Economics, University of Oklahoma

Norman, OK 73019

Tel: 405-325-5492; Fax: 405-325-5842;

e-mail: jdju@ou.edu

March 2002

[1]The author thanks Kate M. Liu for very helpful discussions and constant encouragement. The author has benefited from comments by Steve Ellis, Kala Krishna, Bob Reed, and Daniel Sutter.

Abstract

By showing that increases in the sum of aggregate income and aggregate Marshallian consumer surplus represent potential Pareto improvement, this paper builds a theoretical ground for using aggregate Marshallian consumer surplus as a social welfare indicator. The optimal income distribution that potentially Pareto dominates any other income distribution is studied. It is shown that the income distribution is optimal if and only if the marginal aggregate Marshallian consumer surplus of income is equalized across all consumers. An index that measures income distribution *non-optimality* rather than *inequality* is then developed. It is found that the social cost of income distribution non-optimality in a perfect competitive market could be surprisingly high. Finally, a new approach of social welfare function is developed and it is shown that a social welfare function can be expressed by the sum of aggregate income and aggregate Marshallian consumer surplus if and only if the marginal social welfare of a good is equal to its market price.

Key words: Aggregation; Consumer surplus; Income distribution; Pareto principle; Social welfare

JEL classification: C43; D6; H2

1 Introduction

The aim of this paper is threefold. First, it shows that increases in the sum of aggregate income and aggregate Marshallian consumer surplus represent potential Pareto improvement. The second aim of this paper is to investigate an optimal income distribution that potentially Pareto dominates any other income distribution. The third aim of the paper is to develop a new approach of social welfare function and study conditions under which a social welfare function can be expressed by the sum of aggregate income and aggregate Marshallian consumer surplus.

In an influential open letter to the profession, Harberger (1971) proposes that the Marshallian consumer surplus should be used to measure the individual utility and the social welfare. "Since Harberger's paper, the limitation and pitfalls of consumer's surplus have been demonstrated systematically and definitively by Chipman and Moore among many others" [Slesnick (1998), p. 2159]. However, "consumer's surplus is the overwhelming choice as a welfare indicator" [Slesnick (1998), p. 2110]. For example, Lucas (2000) defines the social cost of inflation as the change in Marshallian consumer surplus of money demand by reducing the interest rate from r to zero.

By showing that increases in the sum of aggregate income and aggregate Marshallian consumer surplus represent potential Pareto improvement, *a theoretical ground is built in this paper for using aggregate Marshallian consumer surplus as a welfare* *indicator.* In a world of many consumers and many goods, state 1 of the economy potentially Pareto dominates state 0 of the economy if consumers at state 1 could at least attain the same utilities as at state 0 through some tax adjustments. The tax adjustments are feasible if the sum of tax revenues is non-negative. Therefore, a positive sum of tax revenues of such adjustments is sufficient for potential Pareto improvement. We show in Section 2 that the sum of multi-step income and commodity tax revenues converges to the change in the sum of aggregate income and aggregate Marshallian consumer surplus. Using multi-step tax adjustments, therefore, we have proved that state 1 potentially Pareto dominates state 0, if the change in the sum of aggregate income and aggregate Marshallian consumer surplus from state 0 to state 1 is positive; and if state 1 Pareto dominates state 0, the change in the sum of aggregate income and aggregate Marshallian consumer surplus from state 0 to state 1 must be non-negative. Thereby, it is established that increases in the sum of aggregate income and aggregate Marshallian consumer surplus represent potential Pareto improvement.

We give conditions in Section 3 under which the aggregate Marshallian consumer surplus is path independent. In contrast to Chipman and Moore's results (1976), the conditions in this paper impose little restriction on consumers' preferences since the conditions are assumed on aggregate demands rather than on individual demands and these conditions are routinely applied by economists. Section 4 discusses the growth of social welfare. When time is varying, the change in the sum of aggregate income and aggregate Marshallian consumer surplus indicates the growth of social welfare. It is shown that the growth rate of social welfare is proportional to the difference between the growth rate of aggregate income and the weighted sum of percentage changes in real prices.

The aggregate Marshallian consumer surplus studied in this paper depends on income distribution rather than on aggregate income. Thus, the sum of aggregate income and aggregate Marshallian consumer surplus provides an ideal technique to study the problem of income distribution. It is shown in Section 5 that the income distribution is optimal if and only if the marginal consumer surplus of income is equalized across all consumers. An index measuring *income distribution non-optimality* rather than *inequality* is then developed.

Adopting the general approach to the measurement of inequality developed by Atkinson (1970), Auerbach and Hassett (2002) have recently developed a new, sophisticated measure of horizontal equity. The new measure and the whole literature that attempts to measure horizontal inequality have been criticized on the grounds that pursuing horizontal equity is in conflict with the Pareto principle (Kaplow 2000 and Kaplow and Shavell 2001).

We develop an index consistent with Pareto principle to measure income distribution *non-optimality* rather than *inequality* in Section 5. As Dalton (1920) noted more than 80 years ago, underlying any measure of equity is some concept of social welfare. Dalton suggested that we should use as a measure of inequality the ratio of the actual level of social welfare to that which would be achieved if income were equally distributed. We modify Dalton's approach and define that the social cost of *income distribution non-optimality* as the difference between the level of aggregate Marshallian consumer surplus that would be achieved if income were optimally distributed and the actual level of aggregate Marshallian consumer surplus. The index that measures income distribution non-optimality (IDN) is defined as the ratio of the social cost of income distribution non-optimality to level of the sum of aggregate income and aggregate Marshallian consumer surplus that would be achieved if income were optimally distributed. If the income is optimally distributed, IDN is zero. As IDN becomes larger, the income distribution is further away from the optimal one. An income distribution with smaller IDN potentially Pareto dominates the income distribution with larger IDN. Therefore, IDN provides a ranking of all income distributions upon the Pareto principle. In a special case that individual demand functions across all consumers are identical, optimal income distribution implies equal income distribution and IDN measures income inequality along the approach suggested by Dalton. Using aggregate Marshallian consumer surplus, social cost of policies, openness index, uniform indices of commodity taxes and income taxes are also developed.

With the well defined social welfare indicators, we then use an example to further explore the problem of income distribution. The result is striking! Suppose that the supply of one good is more elastic than the supply of another good. The maximization of social welfare in the example indicates that more income should be allocated to the consumer who spends more on the good that is more elastically supplied. However, the actual income distribution assumed in the example is misallocated. It turns out that IDN is as high as 22%: the social cost of income misallocation at the perfect competitive equilibrium in the example is about 22% of the maximum level of social welfare and is about 28% of the actual level of social welfare.

Originally formulated by Samuelson (1956), followed by Chipman and Moore (1979), Jorgenson (1997) and many others, the *normative representative consumer* approach represents social welfare as a function of aggregate consumptions which depend on prices and aggregate income that is assumed to be optimally distributed across all consumers (Mas-Colell, Whinston, and Green 1995, p. 118). The problem of income distribution as a major issue of public policy, however, is excluded by the definition of *normative representative consumer* approach.

In Section 6 we develop a new approach of the social welfare function and investigate the conditions under which a social welfare function can be expressed as the sum of aggregate income and aggregate Marshallian consumer surplus. The social welfare is defined as a function of consumption allocations across all consumers rather than aggregate consumptions. Individual consumptions are functions of prices and individual incomes. Therefore, social welfare is represented as a function of prices and income distribution rather than aggregate income. For a given level of aggregate income, different income distributions will give different levels of social welfare in general. It is shown that the change in social welfare along a path of prices and income distribution is equal to the change in the sum of aggregate income and aggregate Marshallian consumer surplus if and only if *the marginal social welfare of a good equals its market price.* This condition is called the *first order condition of the social welfare function.* The condition states that the exchange value of a good in a competitive market is the same regardless of who is consuming; society receives same amount of welfare in the form of money. A social welfare function satisfying the *first order condition of the social welfare function* is defined as the *efficient social welfare function,* which measures the level of *efficient social welfare.* In other words, the sum of aggregate income and aggregate Marshallian consumer surplus is the unique expression of the *efficient social welfare function.*

Classical approaches of welfare economics take postulations on society's behavior as axioms. For example, the *normative representative consumer* approach assumes that the society is represented by a single consumer and the consumer maximizes social utility. The *social choice* approach analyzes the rules to which individual preferences can be aggregated into social preference (Arrow 1963 and Sen 1995 and others). The approach of the *efficient social welfare function* developed in this paper differs from classical approaches in the way that it starts from "what the social value is" rather than "what the social value should be". Correspondingly, a postulation on the *measurement of social value* rather than postulations on the process of aggregating individual utilities into social welfare is taken as the axiom. We argue that the new approach developed in this paper may be more practical and useful. Finally, Section 7 concludes the paper.

2 Potential Pareto Improvement and Aggregate Marshallian Consumer Surplus

There are N goods with positive components representing products and negative components representing factors. There are H consumers in the economy. The price vector is denoted by $p = (p_1, p_2, \cdots, p_N)$ and the income distribution vector is denoted by $y = (y^1, y^2, \cdots, y^H)$. Let c_i^h be the consumption of i^{th} good by the consumer h and $c^h = (c_1^h, c_2^h, \cdots, c_N^h)$ be the consumption bundle for consumer h. A consumption allocation is denoted by a $N \times H$ dimension vector $c = (c^1, c^2, \cdots, c^H)$. $C = (C_1, C_2, \cdots, C_N)$ is the aggregate consumption bundle where $C_i = \sum_{h=1}^{H} c_i^h$. $Y = \sum_{h=1}^{H} y^h$ is the aggregate income. Our focus in this paper is on the distributional issue, behavioral consideration is ignored and the individual's income is assumed to be exogenous.

We will index states by the use of variable, $t \in [0, 1]$. A state of economy is represented by a consumption allocation c or by prices and income distribution (p, y).

For a given level of aggregate income, different income distributions represent different economic states. Suppose that prices and income distribution vary along the path $(p(t), y(t))$ from state 0 to state 1 where $p(t)$ and $y(t)$ are assumed to be differentiable in t. The change in Marshallian consumer surplus for consumer h from state 0 to state 1, $\Delta cs^h(0;1)$, is represented by

$$\begin{aligned}\Delta cs^h(p(0), y^h(0); p(1), y^h(1)) &= -\sum_{i=1}^{N}\int_{p_i(0)}^{p_i(1)} c_i^h(p(t), y^h(t))dp_i(t) \qquad (1)\\ &= -\sum_{i=1}^{N}\int_0^1 c_i^h(p(t), y^h(t))\frac{dp_i(t)}{dt}dt,\end{aligned}$$

and the change in aggregate Marshallian consumer surplus from state 0 to state 1, $\Delta CS(0;1)$, is represented by

$$\begin{aligned}\Delta CS(p(0), y(0); p(1), y(1)) &= \sum_{h=1}^{H}\Delta cs^h(p(0), y^h(0); p(1), y^h(1))\\ &= -\sum_{h=1}^{H}\sum_{i=1}^{N}\int_0^1 c_i^h(p(t), y^h(t))\frac{dp_i(t)}{dt}dt\\ &= -\sum_{i=1}^{N}\int_0^1 C_i(p(t), y(t))\frac{dp_i(t)}{dt}dt \qquad (2)\\ &= -\int_{p(0)}^{p(1)}\sum_{i=1}^{N} C_i(p(t), y(t))dp_i(t) \qquad (3)\end{aligned}$$

Note that the aggregate demand $C_i(p(t), y(t))$ is the function of prices and income distribution rather than aggregate income. Let $\Delta W(0;1)$ be changes in the sum of aggregate income and aggregate Marshallian consumer surplus. That is:

$$\Delta W(0;1) = Y(1) - Y(0) - \sum_{i=1}^{N} \int_0^1 C_i(p(t), y(t)) \frac{dp_i(t)}{dt} dt. \tag{4}$$

$\Delta W(0;1)$ measures the change in social welfare from state 0 to state 1.

We now show that $\Delta W(0;1)$ as a social welfare indicator is consistent with the Pareto principle. That is, state 1 potentially Pareto dominates state 0 if $\Delta W(0;1) > 0$, and $\Delta W(0;1) \geq 0$ if state 1 Pareto dominates state 0.

If consumers at state 1 could face the same prices and income distribution of state 0, $(p(0),\ y(0))$, through M steps of income and commodity tax adjustments, they would make same consumption choices and attain the same utilities as at state 0. Such M steps of tax adjustments are possible if the sum of tax revenues over M steps is non-negative. Therefore, a positive sum of tax revenues over M steps is sufficient for potential Pareto improvement. We want to show that the sum of tax revenues over M step is positive, if the sum of changes in aggregate income and aggregate Marshallian consumer surplus from state 0 to state 1, $\Delta W(0;1)$, is positive. This establishes the Pareto superiority of $c(1)$ over $c(0)$ when $\Delta W(0;1) > 0$.

Let $0 = t_0 < t_1 < \cdots < t_M = 1$ be a partition of state interval $[0,1]$ and $c(t_m)$ is the consumption allocation at state t_m for $m = 1, 2, \cdots, M$. For simplicity, let $t_m = \frac{m}{M}$. Consider a policy at state t_m by government which imposes a specific commodity tax $\tau_c(m)$ and an income tax $\tau_y(m)$ as follows:

$$\begin{aligned} \tau_c(m) &= p(t_{m-1}) - p(t_m) \\ \tau_y(m) &= y(t_m) - y(t_{m-1}) \end{aligned} \quad (5)$$

for $m = 1, 2, \cdots, M$. The equilibrium prices and income distribution at time t_m are $(p(t_m), y(t_m))$ without taxes. However, consumers will face the commodity price $p(t_{m-1})$ and income distribution $y(t_{m-1})$ and are set back to the utility level at time t_{m-1} after $\tau_c(m)$ and $\tau_y(m)$ are imposed. Through M steps of tax adjustments, therefore, consumers are set back from $c(1)$ to $c(0)$.

The tax revenue at t_m is

$$\begin{aligned} R(t_m) &= \sum_{h=1}^{H} y^h(t_m) - y^h(t_{m-1}) + \sum_{h=1}^{H}\sum_{i=1}^{N} \left[p_i(t_{m-1}) - p_i(t_m)\right] c_i^h(t_{m-1}) \\ &= Y(t_m) - Y(t_{m-1}) - \sum_{i=1}^{N} \left[p_i(t_m) - p_i(t_{m-1})\right] C_i(t_{m-1}) \end{aligned} \quad (6)$$

and the sum of tax revenues over M steps is

$$\begin{aligned} TR &= \sum_{m=1}^{M} R(t_m) \\ &= Y(1) - Y(0) - \sum_{i=1}^{N}\sum_{m=1}^{M} \left[p_i(t_m) - p_i(t_{m-1})\right] C_i(t_{m-1}). \end{aligned} \quad (7)$$

As the number of steps approaches infinity, the length of each interval in the partition,

$\Delta t_m = \frac{1}{M}$, approaches zero, which implies that $p_i(t_m) - p_i(t_{m-1})$ will approach zero since $p_i(t)$ is continuous. Using the definition of definite integrals, therefore, we have:

$$\begin{aligned} \lim_{\Delta t_m \to 0} \sum_{m=1}^{M} \left[p_i(t_m) - p_i(t_{m-1}) \right] C_i(t_{m-1}) &= \int_{p_i(0)}^{p_i(1)} C_i(t) dp_i(t) \\ &= \int_0^1 C_i(t) \frac{dp_i(t)}{dt} dt, \end{aligned} \tag{8}$$

and

$$\lim_{\Delta t_m \to 0} TR = Y(1) - Y(0) - \sum_{i=1}^{N} \int_0^1 C_i(t) \frac{dp_i(t)}{dt} dt = \Delta W(0;1) \tag{9}$$

Thus, the sum of tax revenues over M steps, TR, converges to changes in the sum of aggregate income and aggregate Marshallian consumer surplus, $\Delta W(0;1)$.[1] If $\Delta W(0;1)$ is positive, the sum of tax revenues over M steps will be positive as long as M is sufficiently large. This establishes the Pareto superiority of state 1 over state 0 in a weak sense; as explained by Dixit and Norman (1986) and Ju and Krishna (2000a), it will usually be possible to strengthen the result to strict Pareto superiority.

Now consider the reverse problem: if state 1 Pareto dominates state 0, then $\Delta W(0;1)$ must be non-negative. If it were not the case, then $\Delta W(0;1) < 0$, which is equivalent to $\Delta W(1;0) > 0$. This implies that $c(1)$ would be available at state 0 by

[1] If t represents time, a time discount factor will be included in both expressions (2) and (7) and the equality $\lim_{\Delta t \to 0} TR = \Delta W(0,1)$ still holds.

multi-step tax adjustments, while consumers choosing $c(0)$ rather than $c(1)$, which would imply that $c(0)$ Pareto dominates $c(1)$ by the revealed preference. That would contradict the assumption that state 1 Pareto dominates state 0. Summarizing the above, we have:

Proposition 1 *Suppose multi-step tax adjustments are available. If changes in the sum of aggregate income and aggregate Marshallian consumer surplus from state* 0 *to state* 1*,* $\Delta W(0;1)$*, is positive, state* 1 *potentially Pareto dominates state* 0*. If state* 1 *Pareto dominates state* 0*, changes in the sum of aggregate income and aggregate Marshallian consumer surplus from state* 0 *to state* 1*,* $\Delta W(0;1)$*, must be non-negative.*

The above proposition indicates that increases in the sum of aggregate income and aggregate Marshallian consumer surplus represent potential Pareto improvement. The above proposition also indicates that if the sum of aggregate income and aggregate Marshallian consumer surplus is maximized at a consumption allocation c^*, then c^* is optimal in the sense that c^* potentially Pareto dominates any other consumption allocation.

Therefore, the sum of aggregate income and aggregate Marshallian consumer surplus, as a welfare indicator, is in line with Woodrow Wilson:[2]

> "We shall deal with our economic system as it is and as it may be

[2] This quote comes from Feldstein (1976).

modified, not as it might be if we had a clean sheet of paper to write upon; and step by step we shall make it what it should be."

3 Path Dependence Problem

The change in aggregate Marshallian consumer surplus not only depends on prices and income distributions at state 0 and state 1, but also depends on the path connecting two ends. There are several cases; however, $\Delta CS(0;1)$ is uniquely determined by prices and income distributions at two ends.

First, if the state variable t represents time and we evaluate the change in aggregate Marshallian consumer surplus from time 0 to time 1, then $(p(t), y(t))$ is a time path of prices and income distribution, which should be empirically determined. As there is only one time path observed in the real economy, $(p(t), y(t))$ is a unique path and the change in aggregate Marshallian consumer surplus along $(p(t), y(t))$ is single valued.

Second, if income is fixed and only one price p_i is changing, the line integral in (3) is reduced to an integral from $p_i(0)$ to $p_i(1)$ and the path dependence problem does not exist.

Third, in the case that income is fixed at $\overline{y}$ and many prices are changing, rewrite the line integral in (3) as:

$$-\int_{p(0)}^{p(1)} \sum_{i=1}^{N} C_i(p, \overline{y}) dp_i \tag{10}$$

A sufficient condition for the line integral (10) to be path independent is:

$$\frac{\partial C_i(p,\overline{y})}{\partial p_j} = \frac{\partial C_j(p,\overline{y})}{\partial p_i} \text{ for } i \neq j \quad (11)$$

Because the Roy's identity does not hold for aggregate demands, condition (11) imposes little restriction on consumers' preferences and the condition is routinely applied by economists.

Note that we assume income is exogenous in this paper. However, if income is generated from factor supplies, y becomes zero and the condition (11) is again sufficient for path independence.

The above three cases cover a significant portion of economic analyses. Nevertheless, social welfare does depend on adjustment path in some cases. In the field of piecemeal tariff reform, for example, the famous concertina rule states that reducing the highest tariff to the next highest one and then reducing the second highest to the third until all tariffs being reduced to zero will increase the social welfare if all goods are net substitutes. However, if the order is reversed, reducing the lowest tariff to zero, and then reducing the second lowest tariff to zero until all tariffs being removed, will reduce the social welfare.[3]

Path dependence problem has been one of major obstacles to the use of consumer surplus as a social welfare indicator. Chipman and Moore (1976) show that if Roy's identity applies then consumer surplus will be path dependent unless the consumer

[3] For more discussion on piecemeal tariff reform, readers are guided to Ju and Krishna (2000b).

preferences belong to several special types. They conclude, therefore, that consumer surplus is a questionable concept. As we mentioned above, Roy's identity holds for individual demands but not for aggregate demands so that Chipman and Moore's argument does not apply here. Furthermore, we argue that the problem is not whether consumer surplus is path dependent, but rather whether the change of social welfare in the real world depends on the adjustment path. If the adjustment path affects the social welfare in the real world, as generally believed in the field of piecemeal tax reform, then a welfare indicator in theory should reflect rather than throw out this path dependence problem.[4]

4 The Growth of Social Welfare

Let t represent time. The changes in the sum of aggregate income and aggregate Marshallian consumer surplus from state t to state $t + \Delta t$ represent the growth of social welfare. That is:

$$\Delta W(t; t + \Delta t) = Y(t + \Delta t) - Y(t) - \sum_{i=1}^{N} \int_{t}^{t+\Delta t} C_i(p(t), y(t)) \frac{dp_i(t)}{dt} dt \qquad (12)$$

Differentiating the above equality with respect to t and denoting $\frac{dx}{dt}$ as $\dot{x}$ for variable x as usual, we have

[4] A good comparison would be the existence of multiple equilibia, which makes equlibrium analysis richer rather than makes the concept of equlibrium questionable.

$$\dot{W}(t) = \lim_{\Delta t \to 0} \frac{\Delta W(t; t + \Delta t)}{\Delta t} = \dot{Y}(t) - \sum_{i=1}^{N} C_i(t)\dot{p_i}(t). \tag{13}$$

Rewrite the above expression as

$$\left(\frac{W(t)}{Y(t)}\right)\frac{\dot{W}(t)}{W(t)} = \frac{\dot{Y}(t)}{Y(t)} - \sum_{i=1}^{N}\left(\frac{p_i(t)C_i(t)}{Y(t)}\right)\left(\frac{\dot{p_i}(t)}{p_i(t)}\right). \tag{14}$$

The expression (12) is homogeneous of degree one in (p, y); therefore, (p, y) should be normalized and prices and incomes discussed here are real terms. The first term on the right hand side is the growth rate of aggregate income. The second term is the weighted sum of percentage changes in real prices and is called the Divisia price differential.[5]

Noting that $\dot{Y}(t) = C_i(t)\dot{p_i}(t) + \dot{C_i}(t)p_i(t)$, the equation (13) can be written as

$$\dot{W}(t) = \sum_{i=1}^{N} \dot{C_i}(t)p_i(t) \tag{15}$$

The expression (15) reveals that the infinitesimal change in social welfare equals the sum of infinitesimal changes in aggregate consumptions valued on current prices. The expression is a continuous version of the sufficient condition obtained by Ju and Krishna (2000a), which is essentially the revealed preference approach. Summarizing the above, we have:

[5] For more discussion on Divisia price differential, readers are refered to Diewert (2001) and Hillinger (2001).

Proposition 2 *The growth rate of social welfare is proportional to the growth rate of aggregate income minus the Divisia price differential. The infinitesimal change in social welfare equals the sum of infinitesimal changes in aggregate consumptions valued at current prices.*

A positive $C_i(t)$ represents the aggregate demand and a negative $C_i(t)$ represents the aggregate factor supply. The right hand side of (15) is different than the growth in GDP. It is possible that the growth of GDP is positive; however, the growth of social welfare is negative when the consumption of leisure is sufficiently reduced.

5 Optimal Income Distribution and Income Distribution Non-Optimality (IDN) Index

As shown in Section 2, the sum of aggregate income and aggregate Marshallian consumer surplus provides a ranking of all economic states upon the Pareto principle. Using this social welfare indicator, we now discuss the concept of optimal income distribution and then construct an index for income distribution non-optimality (IDN). Applications in the social cost of policies, openness index, uniform indices of commodity taxes and income taxes are discussed as well. An example is given at the end of the section to further illustrate the concepts and shows that *more income should be allocated to the consumer who spends more on the good that is more elastically supplied.*

5.1 Optimal Income Distribution

Let the aggregate income $\overline{Y}$ be fixed and an income distribution be $y(0)$ at state 0. The income distribution is changed to y at state 1. $\sum_{h=1}^{H} y^h(t) = \overline{Y}$ for $0 \leq t \leq 1$. The redistribution of income changes aggregate demands. The equilibrium prices are adjusted correspondingly, which are denoted as $p(t) = p(y(t)) = p(y^1(t), y^2(t), \cdots, y^H(t))$. Using equation (4), changes in the sum of aggregate income and aggregate Marshallian consumer surplus resulting from the income redistribution is:

$$\Delta W(0;1) = \Delta CS(y(0);y) = -\sum_{i=1}^{N} \int_0^1 C_i[p(y(t)), y(t)] \frac{dp_i(y(t))}{dt} dt$$

$$= -\sum_{i=1}^{N} \int_0^1 C_i[p(y(t)), y(t)] \left[\sum_{h=1}^{H} \frac{\partial p_i(y)}{\partial y^h} \frac{dy^h(t)}{dt} \right] dt$$

$$= -\sum_{h=1}^{H} \int_{y^h(0)}^{y^h} \sum_{i=1}^{N} C_i[p(y(t)), y(t)] \frac{\partial p_i(y(t))}{\partial y^h} dy^h(t). \tag{16}$$

Let $Z_h(y) = \sum_{i=1}^{N} C_i[p(y(t)), y(t)] \frac{\partial p_i(y(t))}{\partial y^h}$. The line integrals in (16) do not depend on the path from $y(0)$ to y if $\frac{\partial Z_h(y)}{\partial y^l} = \frac{\partial Z_l(y)}{\partial y^h}$ for $h \neq l$. If $\frac{\partial Z_h(y)}{\partial y^l} \neq \frac{\partial Z_l(y)}{\partial y^h}$, however, different income adjustment paths from $y(0)$ to y may result in different values for ΔCS. We then define $\Delta CS^\wedge$ be the maximized value of the change in aggregate consumer surplus when income distribution is adjusted from $y(0)$ to y along all possible paths. Hence,

$$\Delta CS^{\wedge}(y(0);y) = \max_{y(t)} \left(-\sum_{h=1}^{H} \int_{y^h(0)}^{y^h} \sum_{i=1}^{N} C_i[p(y(t)), y(t)] \frac{\partial p_i(y(t))}{\partial y^h} dy^h(t) \right)$$

subject to

$$\sum_{h=1}^{H} y^h(t) = \overline{Y}$$

Let the solution path of the above maximization problem be denoted as $y^{\wedge}(t)$. We have

$$\Delta CS^{\wedge}(y(0);y) = -\sum_{h=1}^{H} \int_{y^h(0)}^{y^h} \sum_{i=1}^{N} C_i[p(y^{\wedge}(t)), y^{\wedge}(t)] \frac{\partial p_i(y^{\wedge}(t))}{\partial y^h} dy^h(t) \tag{17}$$

In any case, $\Delta CS^{\wedge}(y(0);y)$ is a single valued function of the income distribution y. An optimal income distribution is defined as y^* such that $\Delta CS^{\wedge}(y(0);y)$ is maximized with respect to y. Formally, to find optimal income distribution, we solve the following maximization problem:

$$\max_{y} \ \Delta CS^{\wedge}(y(0);y) \tag{18}$$

subject to

$$\sum_{h=1}^{H} y^h = \overline{Y} \tag{19}$$

Therefore, if optimal income distribution y^* is reached, there does not exist any income redistribution such that the aggregate Marshallian consumer surplus could be improved.

The Lagrangian of the above problem is

$$L = \Delta CS^{\wedge}(y(0); y) + \lambda[\overline{Y} - \sum_{h=1}^{H} y^h]$$

where λ is the Lagrange multiplier and the first order conditions are

$$\frac{\partial \Delta CS^{\wedge}(y(0); y^*)}{\partial y^h} = \frac{\partial \Delta CS^{\wedge}(y(0); y^*)}{\partial y^l} = \lambda \text{ for } h, l = 1, 2, \cdots, H. \tag{20}$$

Summarizing the above, we have:

Proposition 3 *Suppose that* $\Delta CS^{\wedge}(y(0); y)$ *is concave in* y*. For a given level of aggregate income, the aggregate Marshallian consumer surplus is maximized at* y^* *so that the income distribution* y^* *is optimal if and only if the marginal aggregate consumer surplus of income across all consumers is equalized at* y^**.*

5.2 The Income Distribution Non-Optimality (IDN) Index

$\Delta CS^{\wedge}(y(0), y^*)$ measures the difference between the level of aggregate consumer surplus that would be achieved if income were optimally distributed and the actual level of aggregate consumer surplus, therefore represents the *social cost* of income distribution non-optimality. Let the social welfare at income distribution y, $W(y)$, be defined as changes in the sum of aggregate income and aggregate consumer surplus when the income distribution increases from 0 to y. That is:

$$
\begin{aligned}
W(y) &= \Delta W(0,y) = \overline{Y} + \Delta CS^{\wedge}(0,y) \qquad (21) \\
&= \overline{Y} - \sum_{h=1}^{H} \int_0^{y^h} \sum_{i=1}^{N} C_i[p(y^{\wedge}(t)), y^{\wedge}(t)] \frac{\partial p_i(y^{\wedge}(t))}{\partial y^h} dy^h(t).
\end{aligned}
$$

The income distribution non-optimality index (IDN) is defined as the ratio of the social cost of income distribution non-optimality to the level of social welfare that would be achieved if income were optimally distributed. Thus,

$$
IDN = \frac{W(y^*) - W(y(0))}{W(y^*)} = \frac{\Delta CS^{\wedge}(y(0); y^*)}{W(y^*)}. \qquad (22)
$$

It is clear that $0 \leq IDN \leq 1$. If the income is optimally distributed, IDN is zero. As IDN becomes larger, the income distribution is further away from the optimal one. An income distribution with a smaller IDN potentially Pareto dominates the income distribution with a larger IDN. Therefore, the index of IDN provides a ranking of all income distributions upon the Pareto principle.

There are two differences between the income distribution non-optimality index developed above and income inequality indices widely discussed in the literature. First, IDN uses *optimal income distribution* as the status quo while income inequality indices use *equal income distribution* as the status quo. Second, aggregate consumer surplus is used here instead of the unknown social utility functions assumed in income inequality literature.

The measurement of income inequality has been criticized on the grounds that the underlying rationale for pursuing income equity at the expense of individuals' well-being is never stated (Kaplow 2000). Kaplow and Shavell (2001) show that it is possible that an income equity index is improved from state 0 to state 1, while state 0 Pareto dominates state 1. Therefore, they conclude that pursuing income equity may be in conflict with the Pareto principle. The underlying rationale for pursuing income distribution optimality using the IDN index, however, is well defined here: decreases in IDN represent potential Pareto improvement.

Note that if individual demand functions across all consumers are identical, proposition 3 shows that optimal income distribution implies equal income distribution. In this special case, IDN measures income inequality.

5.3 Social Cost of Policies, Openness Index, Uniform Indices of Commodity Taxes and Income Taxes

Using the sum of aggregate income and aggregate Marshallian consumer surplus as the welfare indicator, we now develop several other policy indices. The *social cost of policies, L*, is defined as:

$$L = \Delta W[p(0), y(0); p(1), y(1)]$$

where $(p(0), y(0))$ and $(p(1), y(1))$ are prices and income distribution before and after the policy, respectively. L represents the difference of social welfare before and after the policy.

Similar to the *trade restrictiveness index* developed by Anderson and Neary (1996), the *openness index,* τ_o, is defined as:

$$\Delta W[(p, y); ((1 + \tau_o)q_1, \cdots, (1 + \tau_o)q_N, y)] = 0,$$

where (p, y) are domestic prices and income distribution of a country and $(q_1, \cdots, q_N)$ are world prices. τ_o is the uniform tariff rate that would give the same social welfare as actual trade policies do. The actual trade restrictive policies usually consist of different tariffs and quotas on different goods, which makes trade restrictiveness non-comparable. *Openness index* τ_o, however, is comparable and a larger τ_o indicates more restrictive trade policies.

The *uniform index of commodity taxes,* τ_g, is defined as:

$$\Delta W[(p(1), y); ((1 + \tau_g)p_1(0), \cdots, (1 + \tau_g)p_N(0), y)] = 0,$$

where $(p(0), y)$ and $(p(1), y)$ are prices and income distributions before and after commodity taxes, respectively. τ_g is the uniform commodity tax rate that would give the same social welfare as actual commodity taxes do. The actual commodity taxes on different goods are usually different, which makes the overall level of commodity taxes hard to measure. τ_g, however, provides a ranking of all commodity tax rates and larger τ_g indicates higher commodity tax rates.

The *uniform index of income taxes,* τ_y, is defined as:

$$\Delta W[(p, y(1)); (p, (1+\tau_y)y^1(0), \cdots, (1+\tau_y)y^H(0))] = 0,$$

where $(p, y(0))$ and $(p, y(1))$ are prices and income distributions before and after income taxes, respectively. τ_y is the uniform linear income tax rate that would give the same social welfare as actual income taxes do. Higher τ_y indicates a heavier income tax burden.

5.4 An Example

We now use an example to illustrate the concepts developed above and to further explore the problem of income distribution. Assume that there are two consumers with Cobb-Douglas utility functions and two goods in the market.

$$u^1(c_1^1, c_2^1) = \left(c_1^1\right)^a \left(c_2^1\right)^{1-a}, \; u^2(c_1^2, c_2^2) = \left(c_1^2\right)^b \left(c_2^1\right)^{1-b}, \tag{23}$$

where $1 \geq b > \frac{1}{2} > a \geq 0$. The consumers' demands are

$$c_1^1 = \frac{ay^1}{p_1},\ c_2^1 = \frac{(1-a)y^1}{p_2}, \tag{24}$$

$$c_1^2 = \frac{by^2}{p_1},\ c_2^2 = \frac{(1-b)y^2}{p_2}, \tag{25}$$

and the aggregate demands are

$$\begin{aligned} C_1 &= c_1^1 + c_1^2 = \frac{ay^1 + by^2}{p_1}, \\ C_2 &= c_2^1 + c_2^2 = \frac{(1-a)y^1 + (1-b)y^2}{p_2}. \end{aligned}$$

For simplicity, we assume that the goods are only produced by foreign countries and the foreign supplies are $x_1 = \alpha\left(p_1 - \frac{1}{p_1}\right)$ and $x_2 = \beta\left(p_2 - \frac{1}{p_2}\right)$ for good 1 and good 2, respectively. The total income $\overline{Y}$ is fixed at 1. Let demands equal supplies and the market equilibrium prices are

$$p_1 = \left[\frac{ay^1 + by^2 + \alpha}{\alpha}\right]^{\frac{1}{2}}, \tag{26}$$

$$p_2 = \left[\frac{(1-a)y^1 + (1-b)y^2 + \beta}{\beta}\right]^{\frac{1}{2}}. \tag{27}$$

Let the initial income distribution be $y^1(0) = \frac{9}{98}$ and $y^2(0) = \frac{89}{98}$. Applying (16), the change in aggregate consumer surplus when the income distribution is redistributed from $y(0)$ to y is equal to

$$\begin{aligned}\Delta CS(y(0);y) &= -\{\int_{(\frac{9}{98},\frac{89}{98})}^{(y^1,y^2)} \left(C_1[p(y),y]\frac{\partial p_1(y)}{\partial y^1} + C_2[p(y),y]\frac{\partial p_2(y)}{\partial y^1}\right) dy^1 \\ &\quad + \left(C_1[p(y),y]\frac{\partial p_1(y)}{\partial y^2} + C_2[p(y),y]\frac{\partial p_2(y)}{\partial y^2}\right) dy^2\} \\ &= -\frac{1}{2}\{\int_{(\frac{9}{98},\frac{89}{98})}^{(y^1,y^2)} Z_1(y^1,y^2)dy^1 + Z_2(y^1,y^2)dy^2\}, \qquad (28)\end{aligned}$$

where

$$\begin{aligned}Z_1(y^1,y^2) &= 1 - \frac{a\alpha}{ay^1+by^2+\alpha} - \frac{(1-a)\beta}{(1-a)y^1+(1-b)y^2+\beta} \\ Z_2(y^1,y^2) &= 1 - \frac{b\alpha}{ay^1+by^2+\alpha} - \frac{(1-b)\beta}{(1-a)y^1+(1-b)y^2+\beta}.\end{aligned}$$

It is easy to check that $\frac{\partial Z_1}{\partial y^2} = \frac{\partial Z_2}{\partial y^1}$ so that $\Delta CS(y(0);y)$ is path independent. It can also be verified that $\Delta CS(y(0);y)$ is concave. Using proposition 3, the aggregate consumer surplus is maximized if and only if

$$\begin{aligned}\frac{\partial \Delta CS(y(0);y^{1*},y^{2*})}{\partial y^1} &= \frac{\partial \Delta CS(y(0);y^{1*},y^{2*})}{\partial y^2} \Leftrightarrow \qquad (29) \\ Z_1(y^{1*},y^{2*}) &= Z_2(y^{1*},y^{2*})\end{aligned}$$

Substituting $y^{1*}+y^{2*}=1$ into (29), the optimal income distribution is

$$y^{1*} = \frac{(\alpha+\beta)\,b-\alpha}{(\alpha+\beta)\,(b-a)},\ y^{2*} = \frac{\alpha-(\alpha+\beta)\,a}{(\alpha+\beta)\,(b-a)}. \qquad (30)$$

Let $a = \frac{1}{100}$ and $b = \frac{99}{100}$, which are expenditure shares of good 1 for consumers 1 and 2, respectively. Let $\alpha = \frac{1}{10}$ and $\beta = \frac{9}{10}$, which indicates that the supply of good 2 is more elastic than the supply of good 1. The equation (30) gives the optimal income distribution $y^{1*} = \frac{89}{98}$ and $y^{2*} = \frac{9}{98}$. To maximize the aggregate consumer surplus, more income should be allocated to the consumer who spends more on the good that is more elastically supplied.[6]

Using (21) and (28), the level of social welfare at actual income distribution $y(0)$ is

$$
\begin{aligned}
W(y(0)) &= \overline{Y} + \Delta CS\hat{}\,(0, y(0)) \\
&= 1 - \frac{1}{2}\{\int_{(0,0)}^{(\frac{9}{98},\frac{89}{98})} \left(1 - \frac{0.01 \times 0.1}{0.01y^1 + 0.99y^2 + 0.1} - \frac{0.99 \times 0.9}{0.99y^1 + 0.01y^2 + 0.9}\right) dy^1 \\
&\quad + \left(1 - \frac{0.99 \times 0.1}{0.01y^1 + 0.99y^2 + 0.1} - \frac{0.01 \times 0.9}{0.99y^1 + 0.01y^2 + 0.9}\right) dy^2\} \\
&= 1 - \frac{1}{2}\int_0^{\frac{9}{98}} \left(1 - \frac{0.01 \times 0.1}{0.01y^1 + 0.1} - \frac{0.99 \times 0.9}{0.99y^1 + 0.9}\right) dy^1 \\
&\quad - \frac{1}{2}\int_0^{\frac{89}{98}} \left(1 - \frac{0.99 \times 0.1}{0.01 \times \frac{9}{98} + 0.99y^2 + 0.1} - \frac{0.01 \times 0.9}{0.99 \times \frac{9}{98} + 0.01y^2 + 0.9}\right) dy^2 \\
&= 0.662\,54.
\end{aligned}
$$

The line integral is path independent here and the path of (y^1, y^2), $(0,0) \to (\frac{9}{98}, 0) \to$

[6] At the extreme case, consumer 1 only consumes good 2, and consumer 2 only consumes good 1; the supply of good 1 is fixed and the supply of good 2 is elastic. Consumer 2 will get the same amount of good 1 no matter how little income he or she may have. Therefore, almost all of the total income should be allocated to consumer 1 such that he or she can buy as much good 2 as possible.

$(\frac{9}{98}, \frac{89}{98})$, is chosen to calculate the line integral.

The level of social welfare at the optimal income distribution y^* is

$$\begin{aligned} W(y^*) &= 1 - \frac{1}{2}\{\int_{(0,0)}^{(\frac{89}{98},\frac{9}{98})} \left(1 - \frac{0.01 \times 0.1}{0.01y^1 + 0.99y^2 + 0.1} - \frac{0.99 \times 0.9}{0.99y^1 + 0.01y^2 + 0.9}\right) dy^1 \\ &\quad + \left(1 - \frac{0.99 \times 0.1}{0.01y^1 + 0.99y^2 + 0.1} - \frac{0.01 \times 0.9}{0.99y^1 + 0.01y^2 + 0.9}\right) dy^2\} \\ &= 1 - \frac{1}{2}\int_0^{\frac{89}{98}} \left(1 - \frac{0.01 \times 0.1}{0.01y^1 + 0.1} - \frac{0.99 \times 0.9}{0.99y^1 + 0.9}\right) dy^1 \\ &\quad - \frac{1}{2}\int_0^{\frac{9}{98}} \left(1 - \frac{0.99 \times 0.1}{0.01 \times \frac{89}{98} + 0.99y^2 + 0.1} - \frac{0.01 \times 0.9}{0.99 \times \frac{89}{98} + 0.01y^2 + 0.9}\right) dy^2 \\ &= 0.846\,57, \end{aligned}$$

where the path of $(0,0) \to (\frac{89}{98}, 0) \to (\frac{89}{98}, \frac{9}{98})$ is chosen. Using equation (22), we calculate the income distribution non-optimality index as:

$$IDN = \frac{W(y^*) - W(y(0))}{W(y^*)} = \frac{0.846\,57 - 0.662\,54}{0.846\,57} = 0.217\,38.$$

Comparing to actual social welfare, it gives

$$\frac{W(y^*) - W(y(0))}{W(y(0))} = \frac{0.846\,57 - 0.662\,54}{0.662\,54} = 0.277\,76.$$

Thus, the social cost of income distribution non-optimality is about 22% of the maximum level of social welfare and is about 28% of the actual level of social welfare.

We now use the revealed-preference approach to verify our results.[7] At actual income distribution $y^1(0) = \frac{9}{98}$ and $y^2(0) = \frac{89}{98}$, the equilibrium prices and outputs

[7] Sufficient conditions for potential Pareto improvments using revealed preference approach are

are $(p^1(0), p^2(0)) = (3.1623, 1.0541)$ and $(C_1(0), C_2(0)) = (0.2846, 0.09487)$. At the optimal income distribution $y^{1*} = \frac{89}{98}$ and $y^{2*} = \frac{9}{98}$, the equilibrium prices and outputs are $(p^{1*}, p^{2*}) = (1.4142, 1.4142)$ and $(C_1^*, C_2^*) = (0.07071, 0.6364)$.

$$p^*(C^* - C(0)) = p_1^*(C_1^* - C_1(0)) + p_2^*(C_2^* - C_2(0)) = 0.46335 > 0.$$

Consumers choose C^* while $C(0)$ is feasible; therefore, C^* Pareto dominates $C(0)$.

The example indicates that a competitive efficient equilibrium could be far from optimal. Both $C(0)$ and C^* are competitive efficient. However, different income distributions result in different levels of aggregate demands, which implies different levels of social welfare. These results do not depend on the assumption of exogenous income. Assuming that the high ability consumer spends more on the good that is less elastically supplied, similar results will be obtained when income is endogenously determined.

6 Aggregate Marshallian Consumer Surplus and Efficient Social Welfare Function

By showing that the increase in the sum of aggregate income and aggregate Marshallian consumer surplus represents potential Pareto improvement, we have built

discussed in Ohyama (1972), Dixit and Norman (1986), Grinols and Wong (1991), and Ju and Krishna (2000a).

a theoretical ground for using aggregate Marshallian consumer surplus as a social welfare indicator. The Marshallian consumer surplus is easy to use; the intuition underlying its interpretation as a welfare measurement is transparent; and the data requirements for implementation are minimal. Furthermore, the aggregate Marshallian consumer surplus defined in this paper has a property that it depends on income distribution rather than aggregate income. We have used this property to study the problem of income distribution and got some interesting results.

A fundamental question would be: under what conditions can a social welfare function be expressed as the sum of aggregate income and aggregate Marshallian consumer surplus? That is what we turn into now.

6.1 Efficient Social Welfare Function

The *normative representative consumer* approach (Samuelson 1956, Chipman and Moore 1979, Jorgenson 1997, and many others) represents social welfare as a function of aggregate consumptions which depend on prices and aggregate income that is assumed to be optimally distributed across all consumers. The problem of income distribution as a major issue of public policy, however, is excluded by the definition of *normative representative consumer* approach. Different from the *normative representative consumer* approach, we define social welfare as a function of consumption allocation across all consumers rather than aggregate consumption. That is,

$$V = V(c) = V(c^1, c^2, \cdots, c^H), \tag{31}$$

where V is the level of social welfare and $c = (c^1, c^2, \cdots, c^H)$ is the consumption allocation. Given the aggregate consumption bundle $C = \sum_{h=1}^{H} c^h$, different consumption allocations will generate different levels of social welfare in general. The social welfare function is assumed to be differentiable and monotonic in c.

Let consumers maximize utilities subject to budget constraints. The consumption bundles c^h are represented as demand functions $c^h(p, y^h)$. The corresponding social welfare function becomes:

$$V = V(p, y) = V(c^1(p, y^1), \cdots, c^H(p, y^H)), \tag{32}$$

which depends on market prices and income distribution. The change in social welfare from state 0 to state 1 is

$$\begin{aligned} \Delta V(0;1) &= V(c(1)) - V(c(0)) \\ &= V(p(1), y(1)) - V(p(0), y(0)). \end{aligned}$$

The following proposition gives conditions under which $\Delta V(0;1)$ can be expressed as changes in the sum of aggregate income and aggregate Marshallian consumer surplus from state 0 to state 1, $\Delta W(0;1)$.

Proposition 4 *The change in social welfare along a path of prices and income distribution is equal to the change in the sum of aggregate income and aggregate Marshallian consumer surplus if and only if the marginal social welfare of a good equals its market price. That is:*

$$\frac{\partial V(c)}{\partial c_i^h} = p_i, \text{ for } h = 1, 2, \cdots, H \text{ and } i = 1, 2, \cdots, N \tag{33}$$

Proof: We first prove the sufficient condition. Suppose prices and income distribution vary along the path $(p(t), y(t))$ from state 0 to state1. The change in social welfare from state 0 to state 1 is:

$$\begin{aligned}
& \Delta V(0;1) = V[c(p(1), y(1))] - V[c(p(0), y(0))] \\
= & \sum_{h=1}^{H}\sum_{i=1}^{N}\int_{c_i^h(0)}^{c_i^h(1)} \frac{\partial V(c)}{\partial c_i^h} dc_i^h(t) = \sum_{h=1}^{H}\sum_{i=1}^{N}\int_{c_i^h(0)}^{c_i^h(1)} p_i(t) dc_i^h(t) \qquad (34) \\
= & \sum_{h=1}^{H}\sum_{i=1}^{N}\left[p_i(t)c_i^h(t) \mid_0^1 - \int_{p_i(0)}^{p_i(1)} c_i^h(p(t), y^h(t)) dp_i(t)\right] \\
= & \sum_{h=1}^{H}\left[y^h(1) - y^h(0) - \sum_{i=1}^{N}\int_0^1 c_i^h(p(t), y^h(t))\frac{dp_i(t)}{dt}dt\right] \\
= & Y(1) - Y(0) - \sum_{i=1}^{N}\int_0^1 C_i(p(t), y(t))\frac{dp_i(t)}{dt}dt = \Delta W(0;1)
\end{aligned}$$

where the condition (33) is used to get equality (34).

We now prove the necessary condition. Rewrite the sum of changes in aggregate income and aggregate Marshallian consumer surplus from state 0 to state 1, $\Delta W(0;1)$, as follows:

$$\begin{aligned}
\Delta W(0;1) &= Y(1)-Y(0)-\sum_{i=1}^{N}\int_0^1 C_i(p(t),y(t))dp_i(t) \\
&= Y(1)-Y(0)-\sum_{h=1}^{H}\sum_{i=1}^{N}\int_0^1 c_i^h(p(t),y^h(t))dp_i(t) \\
&= Y(1)-Y(0)-\sum_{h=1}^{H}\sum_{i=1}^{N}\left[p_i(t)c_i^h(t)\mid_0^1-\int_{c_i^h(0)}^{c_i^h(1)} p_i(t)dc_i^h(t)\right] \\
&= \sum_{h=1}^{H}\sum_{i=1}^{N}\int_{c_i^h(0)}^{c_i^h(1)} p_i(t)dc_i^h(t).
\end{aligned}$$

For any consumption allocation c, $V(c)-V(c(0))=\Delta W(c(0);c)$ implies that

$$V(c)=\Delta W(c(0);c)+V(c(0))=\sum_{h=1}^{H}\sum_{i=1}^{N}\int_{c_i^h(0)}^{c_i^h} p_i(t)dc_i^h(t)+V(c(0)),$$

which is monotonic and differentiable in c. Let the market prices corresponding to c be p. Given the path of prices and income distribution $(p(t),y(t))$, note that a small change of $c_i^h(t)$, $\Delta c_i^h(t)$, only affects $\int_{c_i^h(0)}^{c_i^h} p_i(t)dc_i^h(t)$ in the above line integral for any h and i. Therefore, we have:

$$\frac{\partial V(c)}{\partial c_i^h}=\frac{\partial}{\partial c_i^h}\sum_{h=1}^{H}\sum_{i=1}^{N}\int_{c_i^h(0)}^{c_i^h} p_i(t)dc_i^h(t)=p_i \ .$$

That proves the results. **Q.E.D.**

6.2 The First Order Condition of Social Welfare Function

The condition (33) states that the market price and the (marginal) social value of a good are the same regardless of who is consuming, society receives the same amount of welfare in the form of money.[8] The condition (33) is called *the first order condition of social welfare function.* For given market prices and income distribution, we show next that *the first order condition of social welfare function* reflects Pareto efficiency.

Assume that a consumption c_i^h has two values: first, c_i^h provides an *individual value* to the consumer h; and second, consuming c_i^h by the consumer h also provides a *social value* to the society. The individual value of the consumption is measured by the individual's utility, which is represented by a utility function $u^h(c_1^h, c_2^h, \cdots, c_N^h)$. If the consumer is rational, $u^h(.)$ is maximized subject to the budget constraint, which implies that marginal utility is equal to the product of marginal utility of income and the commodity price.

Let the ratio of the social value to the individual value be a^h for consumer h. A social welfare function can be written as a weighted sum of individuals' utilities, that is,

[8] Harberger (1971) proposes that "the competitive demand price for a given unit measures the value of that unit to the *demander.*" Different from his postulate, the condition (33) states that the competitive price for a given unit measures the value of that unit to the *society.*

$$V = \sum_{h=1}^{H} a^h u^h(c^h). \tag{35}$$

We have the following proposition.

Proposition 5 *For given prices and income distribution, let the social welfare function $V = \sum_{h=1}^{H} a^{h*} u^h(c^h)$ be a weighted sum of individuals' utilities, and the weights be $a^{h*} = 1/\lambda^h(p, y^h)$, where $\lambda^h(p, y^h)$ is the consumer h's marginal utility of income. Suppose that the utility functions $u^h (h = 1, \cdots, H)$ are concave and monotonic. The marginal social welfare of a good at the consumption allocation c^* equals the competitive price if and only if the consumption allocation is competitive efficient.*

Proof: We first prove the sufficient condition. Because c^* is competitive efficient, each consumer maximizes the utility on his or her budget set. We have:

$$Du^h(c^{h*}) = \lambda^h p \text{ for } h = 1, 2, \cdots, H.$$

Thus,

$$\frac{\partial V(c^*)}{\partial c_i^h} = a^{h*} \frac{\partial u^h(c^{h*})}{\partial c_i^h} = \frac{1}{\lambda^h}(\lambda^h p_i) = p_i.$$

We now prove the necessary condition. If the competitive price equals the marginal social welfare of a good, then

$$\begin{aligned} p_i &= \frac{\partial V(c^*)}{\partial c_i^h} = \frac{1}{\lambda^h}\frac{\partial u^h(c^{h*})}{\partial c_i^h} \Leftrightarrow \\ \frac{\partial u^h(c^{h*})}{\partial c_i^h} &= \lambda^h p_i. \end{aligned}$$

Hence, c^* is competitive efficient. **Q.E.D.**

When the prices and income distribution are changing, however, the weights a^{h*} are not invariant, and the weighted sum of utilities is no longer a valid form for the social welfare function. Therefore, the competitive efficiency does not ensure that the marginal social welfare of a good equals its market price under the assumption that the social welfare is the weighted sum of utilities.

Classical approaches of welfare economics are based upon the analysis of individual behaviors and the process of aggregating individuals' behaviors into the society's behaviors. Models of individual behaviors have been well developed. However, models on the aggregating process are complicated and not very successful. "The fallacy of composition" probably implies that it is difficult to quantitatively describe the aggregating process.

To study the society, we suggest a new approach: start directly from assumptions on the society rather than assumptions on individuals and an aggregating process. To make assumptions on the society, we suggest to postulate on observations of the society rather than on the ideal behavior of the society. That is, we start from "as it is and as it may be modified", not from "as it might be if we had a clean sheet of paper to write upon"; and "step by step we shall make it what it should be."

Therefore, we take the postulation on the *measurement of social value* that competitive prices reflect the social values of goods as the axiom of our approach. Mathematically, *the first order condition of social welfare function itself* rather than a maximization problem of the weighted sum of utilities is taken as the axiom. Any social welfare function satisfying *the first order condition of social welfare function* is defined as *efficient social welfare function,* which measures the level of *efficient social welfare.* The proposition 4 can now be restated as follows:

Proposition 6 *The change in efficient social welfare along a path of prices and income distribution is equal to the changes in the sum of aggregate income and aggregate Marshallian consumer surplus.*

In the approach of the *efficient social welfare function,* competitive prices reflect the social values of goods; the sum of aggregate income and aggregate Marshallian consumer surplus is a unique expression of the social welfare function; and the increase in *efficient social welfare* indicates potential Pareto improvement.

7 Conclusion

It is very hard to describe a perfect society exactly and quantitatively. It is even harder to have an agreement on what a perfect society should be. The Pareto principle–that no one is worse off and some people are better off implies the improvement of the society–seems to be a unanimously acceptable criterion for social progress. Therefore, developing a social welfare indicator to measure potential Pareto improvement can be a practical and a useful approach for applied economics. As long as we can measure the direction of social progress, we may make the society better. By showing that the increase in the sum of aggregate income and aggregate Marshallian consumer surplus represents potential Pareto improvement, this paper develops such an indicator to measure social progress. Furthermore, this paper shows that the social welfare function can be uniquely expressed by the sum of aggregate income and aggregate Marshallian consumer surplus if we agree that market prices reflect social values of goods.

For a given income distribution, a perfect competitive market is efficient but may not be optimal. The social cost of *income distribution non-optimality* could be surprisingly high. An index that measures *income distribution non-optimality* (IDN) is developed in this paper. IDN provides a ranking of all income distributions upon the Pareto principle. The income is assumed to be exogenous in this paper. The model could be further explored to incorporate labor and capital supplies so that income becomes endogenous, which is certainly an interesting problem and is left for future research.

References

Anderson, James E. and J. Peter Neary (1996), "A New Approach to Evaluating Trade Policy," *Review of Economic Studies,* 63, pp. 107-25.

Arrow, K. J., *Social Choice and Individual Values,* 2nd ed., 1963, New York: Wiley.

Atkinson, Anthony B. (1970), "On the Measurement of Inequality," *Journal of Economic Theory,* 2, pp. 244-263.

Auerbach, Alan J. and Kevin A. Hassett (2002), "A New Measure of Horizontal Equity," *American Economic Review,* Forthcoming.

Chipman, John S. and James Moore (1976), "The Scope of Consumer's Surplus Arguments," In A. M. Tang et. al. eds., *Evolution, Welfare and Time in Economics: Essays in Honor of Nicholas Georgescu-Roegen.* Lexington: Health-Lexington Books, pp. 69-123.

Chipman, John S. and James Moore (1979), "On Social Welfare Functions and the Aggregation of Preferences," *Journal of Economic Theory,* 21, pp. 111-139.

Dalton, H. (1920), "The Measurement of the Inequality of Incomes," *Economic Journal,* 30, pp. 348-361.

Diewert, W. E. (2001), "The Consumer Price Index and Index Number Theory: a Survey," Discussion Paper No. 01-02, Department of Economics, The University of British Columbia, Vancouver.

Dixit, Avinash and Victor Norman (1986), "Gains from Trade without Lump Sum Compensation, " *Journal of International Economics* 21, pp. 111-122.

Feldstein, Martin (1976), "On the Theory of Tax Reform," *Journal of Public Economics,* 6, pp. 77-104.

Grinols, E.L. and K. Wong (1991), "An Exact Measure of Welfare Change," *Canadian Journal of Economics,* 24, pp. 428-449.

Hillinger, C. (2001), "A General Theory of Price and Quantity Aggregation and Welfare Indicators," *SSRN Working Paper.*

Jorgenson, Dale W., Lawrence Lau, and Thomas Stoker (1980), "Welfare Comparison and Exact Aggregation," *American Economic Review,* 70(2), pp. 268-72.

Jorgenson, Dale W. (1997), *Welfare,* vol 2, Cambridge: The MIT Press.

Ju, Jiandong and Kala Krishna (2000a), "Evaluating Trade Reform with Many Consumers," *Canadian Journal of Economics,* 33(3), pp. 787-798.

Ju, Jiandong and Kala Krishna (2000b), "Welfare and Market Access Effects of Piecemeal Tariff Reform," *Journal of International Economics,* 51, pp. 305–316.

Kaplow, Louis (2000), "Horizontal Equity: New Measures, Unclear Principles," *NBER Working Paper* 7649.

Kaplow, Louis and Steven Shavell (2001), "Any Non-Welfarist Method of Policy Assessment Violates the Pareto Principle," *Journal of Political Economics, 109*(2), pp. 281-286.

Lucas, Robert E. Jr. (2000), "Inflation and Welfare," *Econometrica,* 68(2), pp. 247-274.

Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green (1995), *Microeconomic Theory,* Oxford University Press, New York.

Ohyama, M. (1972), "Trade and Welfare in General Equilibrium," *Keio Economic Studies* 9, pp. 37-73.

Samuelson, Paul A. (1956), "Social Indifference Curves, " *Quarterly Journal of Economics,* 70, pp. 1-22.

Sen, Amartya K. (1995), "Rationality and Social Choice," *American Economic Review,* 85(1), pp. 1-24.

Slesnick, Daniel T. "Empirical Approaches to the Measurement of Welfare." *Journal of Economic Literature,* 1998, 36(4), pp. 2108-65.