# In Search of the Liquidity Effect in a Modern Monetary Model: Technical Appendix

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## 1 Introduction

This appendix provides detailed information on the model. It outlines the relevant equations in the model, determines the steady state, and linearizes the model. Information in this appendix provides the necessary information to replicate the simulations of this paper, using the solution methods outlined in King and Watson [1995, 1998].

## 2 The Equilibrium

These equations describe the equilibrium for the households problem. Households are infinitely lived agents who demand consumption and leisure subject to the following cash, time, and budget constraints respectively:

$$P_t c_t = S_t, \tag{1}$$

$$n_t + l_t + h_t = 1,$$

$$M_t = W_t n_t + D_t^c + D_t^f + R_t X_t + R_t(M_{t-1} - S_t).$$

Maximizing household utility subject to these constraints yields the following two Euler equations:

$$\theta(l_t^{-\zeta})/W_t = \beta E_t[R_{t+1}\theta(l_{t+1}^{-\zeta})/W_{t+1}],$$

$$1/(c_t P_t) = R_t\theta(l_t^{-\zeta})/W_t + h_t'(1/S_{t-1})\theta(l_t^{-\zeta}) - \beta E_t[h_{t+1}'(S_{t+1}/S_t^2)\theta(l_{t+1}^{-\zeta})].$$

The time households spend adjusting their portfolio is a function of the change in their "money to spend":

$$h_t = h(S_t/S_{t-1}).$$

These set of equations come from the firms. Firm output, price, labor demand, capital demand and supply, and dividends are separated according to the number of periods since the firm has adjusted its price. For example, $n_{j,t}$ is the labor demand of a firm in period $t$ that last adjusted its price $j$ periods ago. Cost minimization by firms implies the following two factor demand equations:

$$\psi_t \alpha [n_{j,t}/k_{j,t}]^{1-\alpha} = q_t, \tag{2}$$

$$\psi_t (1-\alpha)[k_{j,t}/n_{j,t}]^{\alpha} = W_t/P_t. \tag{3}$$

Since the real wage and user cost of capital are economy wide costs, the real marginal cost and capital-labor ratio will be the same for all firms. The market clearing conditions for capital and labor given the conditional probabilities of price adjustment ($\eta_j$) are:

$$k_t = \sum_{j=0}^{J-1} \omega_{j+1} k_{j,t} \text{ and } n_t = \sum_{j=0}^{J-1} \omega_j n_{j,t}, \tag{4}$$

where

$$\begin{aligned} \omega_1 &= \sum_{j=1}^{J} \eta_j \omega_j \\ \omega_{j+1} &= (1-\eta_j)\omega_j \text{ for } j = 1, ...J-1. \end{aligned}$$

Therefore, the an individual firm's capital-labor ratio is equal to the aggregate capital-labor ratio, $k_{j,t}/n_{j,t} = k_t/n_t$.

The capital supplier must decide on the optimal levels of investment and capital. The respective Euler equations determining these optimal amounts are:

$$[\phi'(i_t/k_t)]\tau_t = \lambda_t P_t R_t,$$

$$\tau_t = \beta E_t[\tau_{t+1}\{(1-\delta) - [\phi'(i_{t+1}/k_{t+1})][i_{t+1}/k_{t+1}] + [\phi(i_{t+1}/k_{t+1})]\} + \lambda_{t+1} P_{t+1} q_{t+1}],$$

where

$$\lambda_t = \theta(l_t^{-\zeta})/W_t.$$

subject to the capital accumulation equation

$$k_{t+1} - k_t = [\phi(i_t/k_t)]k_t - \delta k_t.$$

The profit function for the capital supplier is:

$$D_t^c = P_t q_t k_t - P_t R_t i_t.$$

The production function for the individual firm is:

$$y_{j,t} = (k_{j,t})^{\alpha}(n_{j,t})^{1-\alpha}.$$

When (4) is used to aggregating capital and labor over all firms, the production function becomes:

$$\overline{y}_t = \sum_{j=0}^{J-1} \omega_{j+1} y_{j,t} = (k_t)^{\alpha}(n_t)^{1-\alpha}. \tag{5}$$

Recall the aggregate level of output is not $\sum_{j=0}^{J-1} \omega_{j+1} y_{j,t}$ but is $\left[\sum_{j=0}^{J-1} \omega_{j+1} y_{j,t}^{(\epsilon-1)/\epsilon}\right]^{\epsilon/(\epsilon-1)}$. To relate the linear index implied by the firm production function with the non-linear aggregate index of output, a auxiliary price index is defined as in Yun [1996]:

$$\overline{P}_t = \left[\sum_{j=0}^{J-1} \omega_{j+1} P_{t-j}^{*}{}^{-\varepsilon}\right]^{-1/\varepsilon}.$$

Taking the firms' demand equation:

$$y_{j,t} = \left(\frac{P_{t-j}^*}{P_t}\right)^{-\varepsilon} y_t, \tag{6}$$

and using it to replace $y_{j,t}$ in the linear index, (5), we get:

$$(k_t)^{\alpha}(n_t)^{1-\alpha} = \left(\frac{P_t}{\overline{P}_t}\right)^{\varepsilon} y_t. \tag{7}$$

Firm profits are returned to the shareholders, the households, in the form of dividends:

$$D_{j,t}^f = P_{t-j}^* y_{j,t} - W_t n_{j,t} - P_t q_t k_{j,t}.$$

Using the factor demand equations, (2) and (3), and the firm demand equation, (6), it can be shown that the firm dividends equation can be rewritten as:

$$D_{j,t}^f = \left[P_{t-j}^* - P_t \psi_t\right] \left(\frac{P_{t-j}^*}{P_t}\right)^{-\varepsilon} y_t.$$

The value functions for those firms adjusting their price and those firms who last adjusted their price $j$ periods ago are:

$$v_{j,t} = D_{j,t}^f \lambda_t + \beta E_t[\eta_{j+1} v_{0,t+1} + (1-\eta_{j+1}) v_{j+1,t+1}],$$

for $j = 1, .., J-2$ and

$$v_{J-1,t} = D_{J-1,t}^f \lambda_t + \beta E_t[\eta_J v_{0,t+1}].$$

The next equations are the marginal value functions. Assuming $v_{0,t}$ is differentiable, profit maximizing satisfies the following first order condition when:

$$\frac{\partial v_{0,t}}{\partial P_t^*} = \frac{\partial D_{0,t}^f}{\partial P_t^*} \lambda_t + \beta E_t \left[(1-\eta_{j+1}) \frac{\partial v_{1,t+1}}{\partial P_t^*}\right] = 0,$$

where

$$\frac{\partial D_{j,t}^f}{\partial P_{t-j}^*} = y_t \left[(1-\varepsilon) \left(\frac{P_{t-j}^*}{P_t}\right)^{-\varepsilon} + \varepsilon \psi_t \left(\frac{P_{t-j}^*}{P_t}\right)^{-\varepsilon-1}\right].$$

The marginal value functions for $j = 1, .., J-1$ are:

$$\frac{\partial v_{j,t}}{\partial P^*_{t-j}} = \frac{\partial D^f_{j,t}}{\partial P^*_{t-j}}\lambda_t + \beta E_t \left[(1-\eta_{j+1})\frac{\partial v_{j+1,t+1}}{\partial P^*_{t-j}}\right],$$

for $j = 1, .., J-2$ and

$$\frac{\partial v_{J-1,t}}{\partial P^*_{t-J+1}} = \frac{\partial D^f_{J-1,t}}{\partial P^*_{t-J+1}}\lambda_t.$$

The following are identity equations for output and the price level:

$$y_t = c_t + i_t,$$

$$P_t = \left[\sum_{j=0}^{J-1} \omega_{j+1} {P^*_{t-j}}^{(1-\varepsilon)}\right]^{1/(1-\varepsilon)}.$$

Finally, the following equation equates loan demand and loan supply:

$$P_t i_t = M_{t-1} - S_t + (M_t - M_{t-1}). \tag{8}$$

Substituting in the cash constraint, (1), reduces the banking constraint, (8) to:

$$P_t Y_t = M_t.$$

# 3 The Steady State

These are the steady state equations for the households. Initially, we will assume the gross steady state rate of money growth is $\mu$. The first two equations are the steady state cash and time constraints.

$$Pc = S,$$

$$n + l + h = 1,$$

The steady states for the Euler equations derived from maximizing household utility are:

$$\mu = \beta R.$$

$$1/(cP) = R\theta(l^{-\zeta})/W + h'(\mu/S)\theta(l^{-\zeta}) - \beta h'(\mu/S)\theta(l^{-\zeta}).$$

The steady state for the time spent by households adjusting their portfolio is:

$$h_t = h(S_t\mu/S_{t-1}).$$

Next are the steady state equations for the firms. Since the firm consumption of investment goods and firm capital supply are equal their corresponding aggregate measure, all capital supply and investment in stated in aggregate terms. The first two equations are the steady states of the factor demand equations:

$$\psi\alpha[n_j/k_j]^{1-\alpha} = q,$$

$$\psi(1-\alpha)[k_j/n_j]^{\alpha} = W/P,$$

The steady state equations for optimal investment and capital supply are:

$$[\phi'(i/k)]\tau = \lambda RP,$$

$$\tau = \beta[\tau\{(1-\delta) - [i/k][\phi'(i/k)] + [\phi(i/k)]\} + \lambda qP],$$

where

$$\lambda = \theta(l^{-\zeta})/W.$$

The steady state capital accumulation equation is:

$$\delta = [\phi(i/k)].$$

The steady state profits for the capital supplier are:

$$D_j^c = Pqk - PRi.$$

Recall the linear index function for $y_t$ in (5). The steady state for that equation is:

$$\overline{y} = (k)^{\alpha}(n)^{1-\alpha}.$$

The steady state right hand side of (7) is:

$$\overline{y} = \left(\frac{P^*}{P}\right)^{-\varepsilon} y \sum_{j=0}^{J-1} \omega_{j+1}\mu^{j\epsilon}.$$

Next are the steady state value functions for those firms who last adjusted their price $j$ periods ago:

$$v_j = D_j^f\lambda + \beta(\eta_{j+1}v_{0,} + (1-\eta_{j+1})v_{j+1})],$$

for $j = 1, .., J-2$ and

$$v_{J-1} = D_{J-1}^f\lambda + \beta\eta_J v_0.$$

where

$$D_j^f = \left[\mu^{-j}P^* - P\psi\right]\left(\frac{\mu^{-j}P^*}{P}\right)^{-\varepsilon} y.$$

The steady state equation for the marginal value function of the price adjusting firm is:

$$\frac{\partial v_0}{\partial P^*} = \frac{\partial D_0^f}{\partial P^*}\lambda + \frac{\beta}{\mu}(1-\eta_1)\frac{\partial v_1}{\partial P^*} = 0,$$

where

$$\frac{\partial D_j^f}{\partial P^*} = y\left[(1-\varepsilon)\left(\frac{\mu^{-j}P^*}{P}\right)^{-\varepsilon} + \varepsilon\psi\left(\frac{\mu^{-j}P^*}{P}\right)^{-\varepsilon-1}\right].$$

The steady states for next $j = 1, .., J-2$ marginal value functions are:

$$\frac{\partial v_j}{\partial P^*} = \frac{\partial D_j^f}{\partial P^*}\lambda + \frac{\beta}{\mu}(1-\eta_{j+1})\frac{\partial v_{j+1}}{\partial P^*},$$

and for $j = J-1$

$$\frac{\partial v_{J-1}}{\partial P^*} = \frac{\partial D_{J-1}^f}{\partial P^*}\lambda.$$

The steady state identity equations are:

$$y = c + i,$$

$$P = \left[\sum_{j=0}^{J-1} \omega_{j+1}(\mu^{-j}P^*)^{(1-\varepsilon)}\right]^{1/(1-\varepsilon)},$$

Finally, the steady state equation derived from the banking and cash constraints is:

$$Py = M.$$

# 4 Linearization around the steady state

This model linearizes around the steady state in two different ways. The value and marginal value functions are deviations from trend while the other variables are percent (log) deviation from trend. For example, $dv_{0,t}$ is the deviation of $v_{0,t}$ from trend while $\widehat{l}_t$ is the percent deviation of $l_t$ from trend.

On the households' side, the linearizations of the cash and time constraints are:

$$\widehat{P}_t + \widehat{c}_t = \widehat{S}_t,$$

$$n\widehat{n}_t + l\widehat{l}_t + h\widehat{h}_t = 0,$$

The Euler equations on the household are linearized as follows:

$$-\zeta\widehat{l}_t - \widehat{W}_t = E_t\left(\widehat{R}_{t+1} - \zeta\widehat{l}_{t+1} - \widehat{W}_{t+1}\right),$$

$$\begin{aligned} -\frac{1}{PC}\left[\widehat{P}_t + \widehat{c}_t\right] \quad = \quad & \frac{R\theta(l^{-\zeta})}{W}[\widehat{R}_t - \zeta\widehat{l}_t - \widehat{W}_t] + \frac{h'\mu\theta(l^{-\zeta})}{S}\left[\frac{\mu h''}{h'}(\widehat{S}_t - \widehat{S}_{t-1}) - \widehat{S}_{t-1} - \zeta\widehat{l}_t\right] \\ & -\frac{\beta h'\mu\theta(l^{-\zeta})}{S}\left[\frac{\mu h''}{h'}(\widehat{S}_{t+1} - \widehat{S}_t) + \widehat{S}_{t+1} - 2\widehat{S}_t - \zeta\widehat{l}_{t+1}\right]. \end{aligned}$$

The time cost constraint linearizes as follows:

$$\widehat{h}_t = \frac{\mu h'}{h}[\widehat{S}_t - \widehat{S}_{t-1}].$$

Now on the firm side, the linearized factor demand equations for capital and labor are:

$$\widehat{\psi}_t + (1-\alpha)(\widehat{n}_t - \widehat{k}_t) = \widehat{q}_t,$$
$$\widehat{\psi}_t + \alpha(\widehat{k}_t - \widehat{n}_t) = \widehat{W}_t - \widehat{P}_t.$$

The equations for optimal investment and capital linearize as follows:

$$\frac{[i/k][\phi''(i/k)]}{[\phi'(i/k)]}[\widehat{i}_t - \widehat{k}_t] + \widehat{\tau}_t = \widehat{\lambda}_t + \widehat{P}_t + \widehat{R}_t,$$

$$\begin{aligned}\widehat{\tau}_t &= \beta[(1-\delta) - [i/k][\phi'(i/k)] + [\phi(i/k)]]\widehat{\tau}_{t+1} \\ &\quad -\beta\left[\frac{[i/k][\phi''(i/k)]}{[\phi'(i/k)]}\right][i/k][\phi'(i/k)][\widehat{i}_{t+1} - \widehat{k}_{t+1}] \\ &\quad +\beta\lambda q P/(\tau)[\widehat{\lambda}_{t+1} + \widehat{P}_{t+1} + \widehat{q}_{t+1}],\end{aligned}$$

where

$$\widehat{\lambda}_t = -\zeta\widehat{l}_t - \widehat{W}_t.$$

The linearized capital accumulation equation is:

$$\widehat{k}_{t+1} = [(1-\delta) + [\phi(i/k)] - [i/k][\phi'(i/k)]]\widehat{k}_t + [i/k][\phi'(i/k)]\widehat{i}_t.$$

The linearization of the definition of the index function, $\overline{y}_t$, for $y_t$ and its relation to $y_t$ are as follows

$$\widehat{\overline{y}}_t = \alpha\widehat{k}_t + (1-\alpha)\widehat{n}_t,$$

$$\widehat{\overline{y}}_t = \varepsilon\widehat{P}_t + \widehat{y}_t - \left(\frac{y}{\overline{y}}\right)\left(\frac{P^*}{P}\right)^{-\varepsilon}\sum_{j=0}^{J-1}\omega_{j+1}\varepsilon\mu^{j\varepsilon}\widehat{P}^*_{t-j}.$$

The profit function for the capital supplier is linearized as follows:

$$(1/P)dD^c_t - (D^c/P)\widehat{P}_t = qk[\widehat{q}_t + \widehat{k}_t] - Ri[\widehat{R}_t + \widehat{i}_t].$$

The linearized value function for the firm that last adjusted its price $j$ periods ago is:

$$dv_{j,t} = \lambda dD^f_{j,t} + \lambda D^f_j\widehat{\lambda}_t + \beta(\eta_{j+1}dv_{0,t+1,} + (1-\eta_{j+1})dv_{j+1,t+1})],$$

for $j = 1, .., J-2$ and

$$dv_{J-1,t} = \lambda dD^f_{J-1,t} + \lambda D^f_{J-1}\widehat{\lambda}_t + \beta\eta_J dv_{0,t+1}],$$

where

$$
\begin{aligned}
(1/P)dD^f_{j,t} - (D^f_j/P)\widehat{P}_t \;&=\; y\left[(1-\varepsilon)\left(\frac{\mu^{-j}P^*}{P}\right)^{1-\varepsilon} + \varepsilon\psi\left(\frac{\mu^{-j}P^*}{P}\right)^{-\varepsilon}\right]\left(\widehat{P}^*_{t-j} - \widehat{P_t}\right) \\
&\quad + y\left[\left(\frac{\mu^{-j}P^*}{P}\right)^{1-\varepsilon} - \psi\left(\frac{\mu^{-j}P^*}{P}\right)^{-\varepsilon}\right]\widehat{y}_t - \psi y\left(\frac{\mu^{-j}P^*}{P}\right)^{-\varepsilon}\widehat{\psi}_t.
\end{aligned}
$$

The notation for the marginal value functions, $\frac{\partial v_{j,t}}{\partial P^*_{t-j}}$, is $\mathbf{mv}_{j,t}$. The notation for marginal dividends, $\frac{\partial D^f_{j,t}}{\partial P^*_{t-j}}$, is $\mathbf{mD}_{j,t}$. The linearized equation for the marginal value function of the price adjusting firm is:

$$
\lambda\mathbf{dmD}_{0,t} + \lambda\mathbf{mD}_0\widehat{\lambda}_t = -\frac{\beta}{\mu}(1-\eta_1)\mathbf{dmv}_{1,t+1},
$$

where

$$
\begin{aligned}
\mathbf{dmD}_{j,t} \;&=\; \varepsilon y\left[(1-\varepsilon)\left(\frac{\mu^{-j}P^*}{P}\right)^{-\varepsilon} + (\varepsilon+1)\psi\left(\frac{\mu^{-j}P^*}{P}\right)^{-\varepsilon-1}\right][\widehat{P}_t - \widehat{P}^*_{t-j}] \\
&\quad + y\left[\varepsilon\psi\left(\frac{\mu^{-j}P^*}{P}\right)^{-\varepsilon-1}\right]\widehat{\psi}_t + \mathbf{mD}_j\widehat{y}_t.
\end{aligned}
$$

The linearized equations for next $j = 1, .., J-2$ marginal value functions are:

$$
\mathbf{dmv}_{j,t} = \lambda\mathbf{dmD}_{j,t} + \lambda\mathbf{mD}_j\widehat{\lambda}_t + \frac{\beta}{\mu}(1-\eta_{j+1})\mathbf{dmv}_{j+1,t+1},
$$

and for $j = J-1$ its equation is:

$$
\mathbf{dmv}_{J-1,t} = \lambda\mathbf{dmD}_{J-1,t} + \lambda\mathbf{mD}_{J-1}\widehat{\lambda}_t.
$$

The linearized versions of the identity equations for aggregate output and the price level are:

$$
\widehat{y}_t = \frac{c}{y}\widehat{c}_t + \frac{i}{y}\widehat{i}_t,
$$

$$
\widehat{P}_t = \left[\frac{P^*}{P}\right]^{(1-\varepsilon)}\sum_{j=0}^{J-1}\mu^{j(\varepsilon-1)}\omega_{j+1}\widehat{P}^*_{t-j}.
$$

Finally, linearized equation for the money constraint is:

$$
\widehat{P}_t + \widehat{Y}_t = \widehat{M}_t.
$$

# References

[1] King, Robert G., and Mark Watson, 1995, 'System Reduction and Solution Algorithms for Singular Linear Difference Systems Under Rational Expectations', manuscript, University of Virginia, forthcoming *Computational Economics*.

[2] King, Robert G. and Mark Watson, 1998, 'The Solution of Singular Linear Difference Systems Under Rational Expectations', *International Economic Review*, 39, 1015-1026.

[3] Yun, Tack, 1996, 'Nominal Price Rigidity, Money Supply Endogeneity, and Business Cycles', *Journal of Monetary Economics*, 37, 345-370.