Optimal Product Differentiation in a Circular Model

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Comments welcome

Abstract

Studies employing circular models have typically adopted linear or quadratic transportation cost functions, and firms are assumed to be located equi-distantly on the circle. In this paper, we use a general transportation cost function, and consider a duopoly location-then-price game in a circular model. We characterize a mild sufficient condition to guarantee that equi-distance among firms is an equilibrium feature. In contrast to the commonly held view that (strictly) convex transportation cost function leads to maximum (symmetric) product differentiation (i.e., it is sufficient), we show that it is neither necessary nor sufficient. Instead, it is the concavity of the first derivative of the transportation cost function that guarantees equi-distance in equilibrium. Our welfare results show that when the social planner can choose both location and price, social optimum involves maximum product differentiation. However, equi-distance location may not maximize social welfare in the market equilibrium, or in the social optimum when social planner can only choose location but not price. In the case of commonly used linear or quadratic transportation cost functions, equi-distance leads to maximum consumer surplus and social surplus.

1 Introduction

Since the circular model was introduced in Salop (1979), it has been the workhorse for analyzing spatial competition among differentiated firms. Studies employing circular models often assume that firms are symmetrically (or equi-distantly) located on the circle, whether the number of firms is exogenous or not. In particular, when entry is allowed, it is commonly assumed that after entry

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takes place, firms relocate so that the new number of firms are still equi-distantly located from each other. There have been some studies analyzing the optimal choice of locations in circular models. However, they typically employ one of the following three types of transportation cost functions: (1) linear; (2) quadratic and (3) linear-quadratic.[1] With these specific forms of transportation costs, it is often found that equi-distance does constitute an equilibrium (not necessarily unique). Beyond the symmetric equilibrium, no equilibrium with uneven-spacing has been identified unless additional firm heterogeneity is introduced (e.g., Vogel (2008)) or the format of game is changed (e.g. Gupta et. al. (2004)).

In this paper, we consider a circular model with a general transportation cost function $f(l)$, where l is the distance between the consumer and the firm which he/she buys from. We consider the standard location-then-price two-stage game, where two firms choose locations in the fist stage, and after observing each other's location choice, price decisions are made. We are interested in the following questions: (1) How does location choice affect the intensity of competition? (2) What location will firms choose in the subgame perfect equilibrium? In particular, when will firms choose maximum product differentiation (e.g. one firm is located at 0 while the other at 1/2?).[2] (3) How does location choice affect social and consumer welfare?

To the best of our knowledge, this paper is the first study analyzing optimal location choice with general transportation cost function in a circular model (or Hotelling model). In our model, there is no additional firm heterogeneity beyond the obvious heterogeneity in firms' locations. Our main goal is to characterize conditions on the form of transportation cost $f(l)$, such that equi-distance location constitutes a subgame perfect equilibrium of the location-then-price game.[3] We find that, different from what is commonly thought that (strict) convexity of $f(l)$ is what is needed to obtain the equi-distance result, it is the concavity of the first derivative of $f(l)$ that matters.[4] We also construct examples where the sufficient condition is violated, and equi-distance is not an equilibrium.

We identify two sources of efficiency loss. The first source is termed *price distortion*, which occurs when firms choose different prices and some consumers do not buy their more preferred products. This leads to inefficient switching and lower social surplus. The second source is called

[1]Most studies use one of the first two types of transportation cost functions, which obviously are special cases of the third type.

[2]This is with a little abuse of the term 'maximum,' since whenever one arc between the two firms becomes longer, the other arc necessarily shrinks.

[3]Our usage of general transportation cost function is not without its cost. One major cost is that, in the current model, we can only analyze the competition between two firms. Consequently the current model does not allow us to analyze issues such as entry and merger etc.

[4]In a circular model of two firms, de Frutos et. al. (2002) point out the equivalence between games with convex transportation costs and games with concave transportation costs. They argue that one can analyze convex transportation cost without loss of generality, thus equi-distance is always an equilibrium. We take it one step further in this paper by showing that once one considers general transportation cost functions (as opposed to for example, the linear-quadratic functions), it's not the concavity/convexity of transportation cost function that matters, and convex transportation cost function cannot guarantee equi-distance in equilibrium either.

location distortion. The unique optimal choice of location which minimizes aggregate transportation cost involves equi-distance. Our results show that when the social planner chooses both location and price, social optimum involves maximum product differentiation. However, nonequi-distance may be optimal when the social planner can only choose location, in which case the social planner has to balance between the two types of distortion. We also identify necessary condition under which equi-distance maximizes social welfare, and find that when transportation cost takes the commonly used linear or quadratic forms, equi-distance leads to a subgame perfect equilibrium, which also maximizes both consumer surplus and social surplus.

The rest of the paper is organized as follows. In Section 2 we review the relevant literature. The model is presented in Section 3 and Section 4 contains the analysis. We conclude in Section 5. All proofs can be found in the Appendix.

2 Literature review

There is a large and still growing literature employing circular models. Most studies in this literature assume that there are exogenous number of firms evenly spaced on the circle.[5] With the assumption of symmetric location, circular model provides a convenient way to analyze imperfect competition among $n \geq 2$ firms. This is probably one of the main reasons for the popularity of circular models. To list just a few, circular model has been used to study issues such as advertising (Aghion and Schankerman (2004)), hospital competition (Brekke, Siciliani and Straume (2008)), market segmentation (Coibion, Einav and Hallak (2007), Shapiro and Shi (2008)), competition policy (Grossman and Shapiro (1984)) and collusion (Bae and Choi (2007)). Selten and Apesteguia (2005) conduct experiments on a location game, with linear transportation cost and brands symmetrically distributed on the unit circle. They find that the subjects' behavior is influenced by imitative tendencies and attempts to cooperate.

Within the literature using circular models, the strand most relevant to our paper is the one analyzing location then price games. Anderson (1986) consider two firms and the family of increasing and strictly convex transportation cost functions with $f(l = 0) = 0$. He characterizes the necessary condition for the existence of a subgame perfect equilibrium, when existence of pure strategy price equilibrium are required. Kats (1995) analyzes a circular model of two firms with linear transportation cost. Firm 1's location is normalize to zero. It is shown that there are a contiunuum of equilibria which firms are indifferent, with firm 2's location between 1/4 and 3/4. de Frutos et. al. (1999) analyzes the equilibrium location choice in a model with two firms with a linear-quadratic transportation cost function. They find that when transportation cost function

[5]Circular models have also been employed to analyze issues such as entry and efficiency (e.g. Bharskar and To (2004), Liu and Serfes (2005)). Matsumura and Okamura (2006) investigate the issue of excessive entry when transportation costs are convex, but they assume that after entry firms are equidistantly located on the circle.

takes the form $f(l) = bl - bl^2$ (using our notation), which is concave with same coefficients (but of different signs) of the linear and quadratic terms, there exists a subgame perfect equilibrium where the two firms are evenly spaced on the circle. While the previous studies have considered only two firms and thus have ignored firm entry, Economides (1989) allows entry. In particular, he considers a three stage game with symmetric firms, where firms first decide whether or not to enter, then the location if they enter, and finally choose prices. With quadratic transportation cost (so pure strategy price equilibrium always exists for any location combinations), he finds that there is a subgame perfect Nash equilibrium where firms are evenly spaced on the circle. While this equilibrium is not necessarily unique, no equilibrium with un-evenly spaced firms is characterized or shown to exist.

It has been show that certain changes toward the standard location-then-price games can lead to equilibrium with asymmetric locations. Intuitively, one can obtain equilibrium with asymmetric locations by making firms asymmetric. For example, Vogel (2008) introduces firm heterogeneity in marginal cost into the model so that some firms are more productive (lower marginal costs) than others. In this asymmetric firms setting, with linear transportation cost, he finds that more productive firms will be more isolated in the equilibrium. Another way to obtain asymmetric location is to change the game format. Gupta et. al. show that, in a location-then-quantity (instead of location-then-price) game with symmetric firms, the equidistant location pattern is only one of many equilibria, and non-equidistant equilibrium may arise. However, as long as firms are symmetric, equi-distant location is always an equilibrium, regardless of whether it is the unique one. In our paper, we find that even with symmetric firms, once general transportation cost function is considered, the symmetric location may not be an equilibrium any more.

Location choice has also been studied in Hotelling models. Hotelling (1929) explains that each firm has incentive to move toward the middle (principle of minimal differentiation). d'Aspremont et. al. (1979) point out the flaw in Hotelling's argument, and show that there is actual maximal differentiation when transportation cost is quadratic. Economides (1986) considers transportation cost function of the form $f(l) = l^\alpha$ (again using our notation), where $\alpha \in (1, 2)$. He finds that there exists $\bar{\alpha} \approx 1.26$ such that when $\alpha < \bar{\alpha}$, there is no equilibrium in locations when only pure strategy price equilibrium is considered. However, when $\alpha > \bar{\alpha}$, equilibrium in location exists with two firms located equally away from the middle of the Hotelling line. Once mixed strategy in price is considered, Osborne and Pitchik (1987) finds that with linear transportation cost, there is a unique equilibrium in location with two firms' locations symmetric about 1/2.

3 The model

Two firms, each with constant marginal cost $c \geq 0$, choose where to enter in the market, represented by a unit circle. There is a continuum of consumers of measure one uniformly distributed on the circle. Each consumer is identified by his location on the circle, which corresponds to his ideal brand. Consumers buy either one unit of product from either firm and derive a utility of V, or do not buy at all and enjoy a utility of zero. When a consumer buys from a firm away from his ideal location, there is a disutility in the form of transportation cost $f(l)$, where l is the closest distance between the consumer and the firm traveling along the circle. We assume that $f(l = 0) = 0$, $f'(l) > 0$ for $l \in [0, 1/2]$.[6] Without loss of generality, we assume that firm 1 is located at 0, and firm 2 is located at $L \leq \frac{1}{2}$. Location is counted clockwise. For any consumer located at x, we can then calculate his utility when buying from either firm as the following,

$$U_1 = \begin{cases} V - p_1 - f(x), & \text{if } x \in [0, 1/2], \\ V - p_1 - f(1-x) & \text{if } x > 1/2. \end{cases}$$

$$U_2 = \begin{cases} V - p_2 - f(L-x), & \text{if } x \in [0, L], \\ V - p_2 - f(x-L), & \text{if } x \in (L, 1/2 + L], \\ V - p_2 - f(1-x+L) & \text{if } x > 1/2 + L. \end{cases}$$

The case of $x \in [0, L]$ is plotted in Figure 1. We assume that V is sufficiently high, ensuring that consumers will buy in the equilibrium.

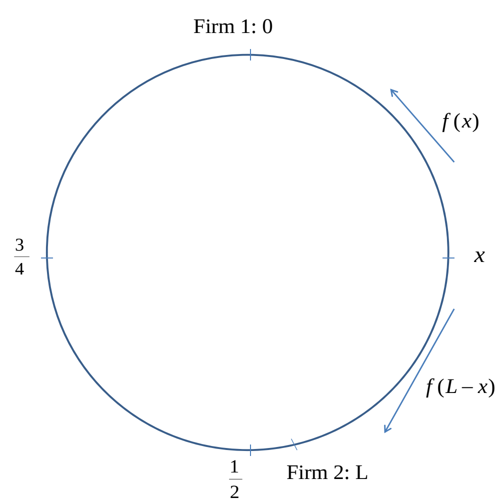

Figure 1: Firm locations and transportation costs

We analyze the following two stage game:

- Stage 1: *Location choice stage.* Firms make their location choices. Due to symmetry, we

[6]On a unit circle, the nearest distance between any two locations on the circle cannot be more than 1/2.

normalize firm 1's location to be zero, and denote firm 2's location by $L \leq 1/2$.[7]

- Stage 2: *Pricing stage.* Firms simultaneously and independently set their prices. When L is small and firms are close to each other, there may not exist a pure strategy equilibrium in prices, and we have to look at the mixed strategy equilibrium.

We look for a symmetric subgame perfect Nash equilibrium, represented by firms' optimal pricing strategies $p_1^*(L) = p_2^*(L)$ in stage 2, and firm 2's optimal location L^* in stage 1.

4 Analysis

We solve the game backwards, starting with the pricing stage.

4.1 Stage 2: Pricing decisions

The circle can be divided into two arcs in between the two firms. The (weakly) smaller arc is the part from 0 to L. The (weakly) larger arc is from L to 1 (or 0). Let x_1 and x_2 denote the location of the marginal consumers on the smaller and larger arc respectively, and let m and n denote firm 1's corresponding market shares. Then it must be that $x_1 = m$ and $x_2 = 1 - n$. Marginal consumers are indifferent from buying from either firm, thus[8]

$$\begin{aligned} x_1 &: \quad V - p_1 - f(m) = V - p_2 - f(L - m) \\ x_2 &: \quad V - p_1 - f(n) = V - p_2 - f(1 - L - n) \end{aligned} \tag{1}$$

The solution to these two equations leads to two implicit functions of firm 1's market shares $m = m(L, p_1, p_2)$ and $n = n(L, p_1, p_2)$. Then firms' profits are:

$$\pi_1 = (p_1 - c)(m + n), \quad \pi_2 = (p_2 - c)(1 - m - n).$$

Firm i's problem is,

$$\max_{p_i} \pi_i(L, p_1, p_2), \quad i = 1, 2.$$

Next we search for the candidate for the symmetric pure strategy equilibrium in prices. This equilibrium candidate is summarized in Lemma 1.

[7] In this case, simultaneous entry game is equivalent to the sequential entry game, due to normalization of firm 1's location.

[8] An implicit assumption made is that the shorter routes for marginal consumer at x_2 to travel to firms are clockwise to firm 1 and counter clockwise to firm 2. This clearly holds in a symmetric equilibrium ($p_1 = p_2$), but needs to be checked when firms deviate.

Lemma 1 *The symmetric pure strategy equilibrium candidate is characterized by the following.*

i) Each firm chooses a price of: $p^*(L) = c + \frac{1}{\frac{1}{f'(\frac{L}{2})} + \frac{1}{f'(\frac{1-L}{2})}}$*;*

ii) Firms split the market equally on each arc: $m = L/2$*,* $n = (1-L)/2$ *and* $m+n = 1-m-n = \frac{1}{2}$*.*

iii) Each firm enjoys a profit of $\pi^*(L) = \frac{1}{\frac{2}{f'(\frac{L}{2})} + \frac{2}{f'(\frac{1-L}{2})}}$*.*

Proof. See Appendix. ■

Note that this is the unique candidate for a symmetric pure strategy equilibrium. Next we want to characterize the conditions under which this equilibrium candidate is indeed an equilibrium. We are particularly interested in the case of maximum product differentiation ($L = 1/2$). When $L = 1/2$, the prices in Lemma 1 are $p^*(L = 1/2) = c + \frac{f'(\frac{1}{4})}{2}$. Next we will provide conditions which will govern whether (p^*, p^*) constitutes an equilibrium in stage 2, and how $\pi^*(L = 1/2$ compares with profits when other strategies are played.

Assumption 1: The function

$$g(l) \equiv l\,[f(1/2 - l) - f(l)]$$

is concave in l, i.e. $g''(l) \leq 0$, $\forall l \in [0, 1/2]$.

Assumption 2: $f'(l)$ is concave in l, $\forall l \in [0, 1/2]$.

Proposition 1 *If Assumption 1 is satisfied, then* $p_i = p^*(L = 1/2) = c + \frac{1}{2} f'(\frac{1}{4})$ $(i = 1, 2)$ *constitutes an equilibrium in the pricing stage for* $L = 1/2$*.*

Proof. See Appendix. ■

Note that Assumption 1 is a sufficient condition for the equilibrium candidate to be an equilibrium when $L = 1/2$. It is also the unique symmetric pure strategy equilibrium. It is easy to see that a necessary condition is $g''(l = 1/4) \leq 0$. If the equilibrium candidate in Lemma 1 is not an equilibrium, then there is no symmetric pure strategy equilibrium. Due to symmetry, asymmetric equilibria (pure or mixed) must be in pairs and thus there are even numbers of such equilibria. Since the number of all equilibria is odd, there must be a symmetric mixed strategy equilibrium. In this equilibrium, prices must be bounded from both above (by consumer's reservation utility V) and below (by marginal cost c). Suppose that both firms choose prices according to a CDF $F(p)$ on the interval $[\underline{p}, \bar{p}]$. Let $\tilde{p}$ denote this mixed strategy played by each firm, and let $\pi^{mixed}(L)$ denotes each firm's corresponding (expected) profit. Next, we want to see how $\pi^*(L = 1/2)$ compares with (1) $\pi^*(L \neq 1/2)$ and (2) $\pi^{mixed}(L)$, where $\pi^*(L)$ is as provided in Lemma 1. The results are summarized in the next Proposition.

Proposition 2 *If Assumption 2 holds, then*

(i) $\pi^*(L = 1/2) \geq \pi^*(\tilde{L}), \forall \tilde{L} \neq 1/2.$

(ii) $\pi^*(L) > \pi^{mixed}(L), \forall L.$

Proof. See Appendix. ■

From Proposition 2, (i) implies that $\pi^*(L)$, or firm's profit under the symmetric pure strategy equilibrium candidates reaches its maximum at $L = 1/2$. Combining it with (ii), together they imply that each firm's profit is still maximized at $\pi^*(L = 1/2)$, even when symmetric mixed strategy equilibria are considered.

4.2 Stage 1: Location choices

Theorem 1 *When Assumption 1 and 2 are satisfied,* $L = 1/2$ *with the corresponding strategies given in Lemma* 1 *constitutes a subgame perfect Nash equilibrium.*[9]

Proof. From Proposition 1, we know that if $g''(l) \leq 0$, then $p_1 = p_2 = p^*(L = 1/2)$ constitutes a symmetric pure strategy equilibrium when $L = 1/2$. The corresponding profit is $\pi^*(L = 1/2)$. Proposition 2 shows that if $f'(l)$ is concave in l, then $\pi^*(L)$ is maximized at $L = 1/2$ and $\pi^*(L) > \pi^{mixed}(L)$. Therefore, $L = 1/2$ is optimal choice for firm 2. ■

Corollary 1 *If the transportation cost function is linear, i.e.,* $f(l) = al$ *where* $a > 0$ *is a constant, then* $\forall L \in [1/4, 3/4]$ *is an optimal choice.*

If we set $a = 1$, then this is the same result as in Kats. Next we explain that having $a \neq 1$ does not change this result. When transportation cost is linear, $f'(l)$ is a constant. From Proposition 1, we know that in the symmetric pure strategy equilibrium candidate, prices and profits do not vary with L. Profits from symmetric mixed strategy equilibrium leads to lower profits by Proposition 2. Consequently, any location choice of L is an equilibrium, so long as the corresponding pricing stage has a symmetric pure strategy equilibrium as in 1. It can be shown that when $L \in [1/4, 3/4]$, neither firm has incentive to deviate. Therefore, $\forall L \in [1/4, 3/4]$ is an optimal choice.

Intuitively, one may think that firms would choose maximal product differentiation when the transportation cost function is (strictly) convex, but not when it is (strictly) concave. We have

[9] Note that these two conditions are sufficient but not necessary conditions. First, we have used sufficient conditions for each step. Moreover, technically, we did not rule out the possibility that there is no PSNE in prices for each $L \in (0, 1/2]$, but the profit from the MSNE is maximized when $L = 1/2$.

shown that this intuition is inaccurate.[10] It is the first derivative of the transportation cost function which enters into prices and profits (as in Lemma 1), and Proposition 2 confirms that it is the concavity of this first derivative which gives firms incentive to select maximal product differentiation. Next, to help illustrate this result, we will provide several examples.

Example 1: $f(l)$ is concave but $L = 1/2$ is optimal.

In this example, we select $f(l) = -l^2 + l$.[11] Then

$$g(l) \equiv l\,[f(1/2 - l) - f(l)] = \frac{1}{4}l(1 - 4l).$$

Its second derivative is

$$g''(l) = -2,$$

and Assumption 1 is satisfied.

Moreover, we have

$$f'(l) = 1 - 2l,$$

which is (weakly) concave in l. Thus Assumption 2 is satisfied as well. Thus $L = 1/2$ is optimal.

Note that from Lemma 1, it can be shown that

$$\pi^*(L) = \frac{1}{2}L(1 - L).$$

It is strictly concave in L and the global maximum is achieved at $L = 1/2$.

Example 2: $f(l)$ is convex but $L = 1/2$ is not optimal.

We choose $f(l) = l^3 + 3l$. It can be shown that

$$g(l) = \frac{1}{8}l(1 - 4l)(4l^2 - 2l + 13),$$

and

$$g''(l) = -24l^2 + 9l - 27/2 < 0.$$

Then at $L = 1/2$, the prices given in Lemma 1 is an equilibrium. We then numerically verify that around $L \to \frac{1}{2}$, the corresponding prices also constitute an equilibrium. Profit from the pure strategy equilibrium is

$$\pi^*(L) = \frac{1}{\frac{2}{f'(\frac{L}{2})} + \frac{2}{f'(\frac{1-L}{2})}} = \frac{3}{8}\frac{(L^2 + 4)(5 - 2L + L^2)}{(9 - 2L + 2L^2)}.$$

[10] de Frutos et. al. (1999) point out the equivalence result between concave and convex transportation cost, and one can consider only convex transportation cost function without loss of generality. Therefore, they argue that in the circular model the unique pattern of product differentiation is always maximal differentiation. However, they only consider quadratic functions of transportation cost, and our results show that in general strict convexity of transportation cost function does not ensure maximal differentiation. For details, see our Example 2 next.

[11] de Frutos et. al. (1999) showed that with a transportation cost function of the form $C(z) = bz - bz^2$ leads to maximal differentiation.

It can be shown that

$$\frac{\partial^2 \pi^*(L)}{\partial L^2}|_{L=1/2} = \frac{39}{136} > 0.$$

Therefore, $L = 1/2$ is a local minimum and not optimal.

While we know that $L = 1/2$ is not optimal, it's unclear what the optimal choice of L is. In general, since firm 2 chooses $L \in (0, 1/2]$ to maximizes its profit, there must either exist an optimal choice of L^* which maximizes firm 2's profit, or in the case when firm 2's profit (when switching between pure and mixed strategy equilibrium) is discontinuous in L, an L^* such that firm 2's profit is arbitrarily close to

$$\lim_{L \to L^*} \pi_2(L).$$

4.3 Welfare analysis

Since prices are just transfer between consumers and firms, they do not enter into social surplus, which is equal to U minus aggregate transportation costs. We start with the case of social optimum, where the social planner chooses firm location, but may or may not choose prices.

Social optimum

There are two sources of efficiency loss in the model. First, consumers may not buy from the closer firm (inefficient switching). This happens when firms' prices are different, for example, when there is no pure strategy price equilibrium and firms mix prices. We call this distortion (due to unequal prices) the *price distortion.* Second, even if firms' prices are the same so that there is no inefficient switching, firms' location choice may not be "optimal." For example, in the market equilibrium, firms choose location to maximize profits not social welfare. We call this the *location distortion.* Now we consider how a social planner would choose locations. The results are summarized in Lemma 2.

Lemma 2 *When the social planner can choose both location and prices, the first best outcome is to have prices exogenously equated ($p_1 = p_2$) and $L = 1/2$.*

It is more plausible that while a regulator can control firm location, it cannot control prices. Next we consider the second best, where the social planner chooses location, then the firms choose prices strategically. If there is a symmetric pure strategy price equilibrium when $L = 1/2$ so that the endogenous prices will still be equated, then the social planner will choose $L = 1/2$. In this case, the second best where the regulator can choose location but not prices, is equivalent (in terms of social welfare) to the first best where both location and prices are exogenously chosen. However, if the equilibrium candidate identified in Lemma 1) is not an equilibrium, then the social planner has to weigh the two types of distortions. The optimal location in the second best may involve $L < 1/2$, if it leads to less price distortion (although more location distortion).

Market equilibrium

In any market equilibrium, efficiency loss depends on firms' location (L) and the corresponding equilibrium in the pricing game. When Assumptions 1 and 2 hold, $L = 1/2$ and there is symmetric pure strategy price equilibrium. In this case, market equilibrium also leads to the lowest efficiency loss, which is the same as in the first best. However, whenever $L \neq 1/2$, or the corresponding equilibrium in the pricing game does not involve equal prices, market equilibrium entails more efficiency loss, relative to the first best. Moreover, market equilibrium can do no better than the second best, since in latter, the social planner can mimic the location choice in the market in equilibrium.

Next we analyze how firm location affects consumer surplus in the market equilibrium. We will restrict out attention to symmetric pure strategies in the pricing game. Let $trans(L)$ denote the aggregate transportation cost when firm 2 is located at L, the total consumer surplus is

$$\begin{aligned} CS(L) &= U - p^*(L) - trans(L) \\ &= U - c - \frac{1}{\frac{1}{f'(\frac{L}{2})} + \frac{1}{f'(\frac{1-L}{2})}} - 2\int_{l=0}^{\frac{L}{2}} f(l)dl - 2\int_{l=0}^{\frac{1-L}{2}} f(l)dl \end{aligned}$$

It's easy to see that both $p^*(L)$ and $trans(L)$ are symmetric about $L = 1/2$, that is, $p^*(1/2-\epsilon) = p^*(1/2+\epsilon)$ and $trans((1/2-\epsilon) = trans(1/2+\epsilon)$. Then $CS(L)$ is also symmetric about $L = 1/2$, thus

$$CS'(L)|_{L=\frac{1}{2}} = 0.$$

This implies that $L = 1/2$ is either a local maximum or a local minimum for $CS(L)$. Next, we calculate the second derivative of $CS(L)$ and evaluate it at $L = 1/2$.[12] It can be shown that

$$\frac{d^2CS(L)}{dL^2}|_{L=\frac{1}{2}} = \frac{[f''(1/4)]^2}{4f'(1/4)} - \frac{1}{8}f'''(1/4) - f'(1/4).$$

If this is positive, then $L = 1/2$ is a local minimum. Otherwise, $L = 1/2$ is a local maximum. For equi-distance ($L = 1/2$) to maximize consumer surplus, the necessary condition is $\frac{d^2CS(L)}{dL^2}|_{L=\frac{1}{2}} \leq 0$. Next we look at for the common forms of transportation cost functions. The results are summarized in Lemma 3.

Lemma 3 *When transportation cost is either linear or quadratic, equi-distance ($L = 1/2$) leads to the highest level of consumer surplus.*

[12] The general expression of $\frac{d^2CS(L)}{dL^2}$ is quite tedious without evaluating it at $L = 1/2$. Later we derive the second derivative for specific forms of transportation cost function.

Proof. See Appendix. ■

Note that when the transportation cost function is either linear or quadratic, $L = 1/2$ is an optimal choice and the corresponding pricing game has a symmetric pure strategy equilibrium. Therefore, the equi-distance among firms maximizes social welfare as well.

5 Conclusion

Location choice has been studied extensively in the literature. In circular models of entry, it is commonly assumed that after entry takes place, firms relocate so that they are symmetrically spaced on the circle, and the commonly used forms of transportation cost function are either linear or quadratic. In this paper, we employ a circular model and analyze a location-then-price game with general transportation cost function. We identify a mild sufficient condition on the transportation cost function so that location choice with equi-distance constitutes a subgame perfect equilibrium. Various commonly studied forms of transportation cost function satisfy this condition. Our welfare results show that in the first-best where the social planner can choose location and prices, social optimum involves equi-distance. However, equi-distance does not necessarily maximize social welfare in the market equilibrium, or in the second-best when the social planner can choose location but not prices. When transportation cost function is either linear or quadratic, equi-distance among firms is an equilibrium feature, which leads to maximal consumer surplus and social surplus.

In general, to decide whether equilibrium location exhibits equi-distance, two factors are important: the functional form of transportation cost and the consumer distribution function. In this paper, we have shown that, by relaxing the former (i.e., with a general transportation function), maximum production differentiation may not be an equilibrium. A natural question then is, whether relaxing the latter (consumer distribution) would lead to similar results.[13] This analysis is beyond the scope of this paper, and we reserve it for future research.[14]

Appendix

Proof of Lemma 1

[13]Both the Hotelling and circular model literature typically assume that consumers are uniformly distributed on the line or circle. An exception is Anderson, Goeree and Ramar (1997), who model a location-then-price model with general log-concave consumer densities.

[14]With non-uniform distribution, firm 1 cannot be assumed to be located at zero without loss of generality, and the equilibrium may not be symmetric, making the analysis intractable in a circular type of model. A Hotelling type of model may be more suitable.

We first derive the partial derivatives of m and n with respect to p_1 and p_2. From equation (1), we have

$$V - p_1 - f(m) = V - p_2 - f(L - m)$$

$$\Rightarrow p_1 + f(m) = p_2 + f(L - m).$$

Taking derivative with respect to p_1 at both sides, we can obtain

$$1 + f'(m)\frac{\partial m}{\partial p_1} = -f'(L - m)\frac{\partial m}{\partial p_1}.$$

This implies that

$$\frac{\partial m}{\partial p_1} = -\frac{1}{f'(m) + f'(L - m)}.$$

Similarly we can obtain

$$\frac{\partial n}{\partial p_1} = -\frac{1}{f'(n) + f'(1 - L - n)} \text{ and,}$$

$$\frac{\partial m}{\partial p_2} = -\frac{\partial m}{\partial p_1}, \quad \frac{\partial n}{\partial p_2} = -\frac{\partial n}{\partial p_1}.$$

First order conditions imply

$$\frac{\partial \pi_1}{\partial p_1} = m + n + (p_1 - c)\left(\frac{\partial m}{\partial p_1} + \frac{\partial n}{\partial p_1}\right) = 0 \Rightarrow$$

$$m + n = (p_1 - c)\left(\frac{1}{f'(m) + f'(L - m)} + \frac{1}{f'(n) + f'(1 - L - n)}\right). \tag{2}$$

Similarly, from $\frac{\partial \pi_2}{\partial p_2} = 0$, we can obtain

$$1 - m - n = (p_2 - c)\left(\frac{1}{f'(m) + f'(L - m)} + \frac{1}{f'(n) + f'(1 - L - n)}\right). \tag{3}$$

Next we impose the symmetry condition $p_1 = p_2$. Then $m = L/2$, $n = (1-L)/2$ and $m+n = 1/2$. Plug them into equation (2) or (3), we can obtain

$$p_1^*(L) = p_2^*(L) = c + \frac{1}{\frac{1}{f'(\frac{L}{2})} + \frac{1}{f'(\frac{1-L}{2})}}. \tag{4}$$

Then firms' profits are

$$\pi_i^*(L) = (p_i^*(L) - c) \times \frac{1}{2} = \frac{1}{\frac{2}{f'(\frac{L}{2})} + \frac{2}{f'(\frac{1-L}{2})}}.$$

■

Proof of Proposition 1

Due to symmetry, we only check firm 1's deviations. We fix $p_2^* = c + \frac{1}{2}f'(\frac{1}{4})$, and check whether firm 1 can increase its profit by deviating p_1. Since $L = 1/2$, the two arcs are symmetric and firm 1's market share on each arc must be the same ($m = n$) for any p_1. Moreover, the marginal consumer on the arc from 0 to 1/2 is located at $x = m$. Therefore

$$p_1 + f(m) = p_2^* + f(1/2 - m) \Rightarrow p_1 = p_2^* - f(m) + f(1/2 - m)$$
$$\Rightarrow p_1 = c + \frac{1}{2}f'(1/4) - f(m) + f(1/2 - m).$$

Firm 1's deviation profit is

$$\pi_1(p_1) = (p_1 - c)(m + n) = 2m(p_1 - c).$$

Since there is a one-to-one correspondence between m and p_1, we will express firm 1's profit as a function of m (instead of p_1). Then firm 1's problem is,

$$\max_{m \in (0,1/2]} \pi_1(m) = m\left[f'(1/4) - 2f(m) + 2f(1/2 - m)\right].$$

Take derivative with respect to m, we can obtain

$$\pi_1'(m) = f'(1/4) - 2f(m) + 2f(1/2 - m) - 2mf'(m) - 2mf'(1/2 - m).$$

It's easy to see that $\pi_1'(m = 1/4) = 0$.

The second derivative is

$$\begin{aligned}\pi_1''(m) &= -2f'(m) - 2f'(1/2 - m) - 2f'(m) - 2mf''(m) - 2f'(1/2 - m) + 2mf''(1/2 - m)\\ &= 2mf''(1/2 - m) - 2mf''(m) - 4f'(m) - 4f'(1/2 - m)\end{aligned}$$

It can be shown that the following two statements are equivalent: (i) $\pi_1''(m) \leq 0$, (ii) $g(l) \equiv l\left[f(1/2 - l) - f(l)\right]$ is concave in l (i.e., $g''(l) \leq 0$). Note that $\pi_1'(m = 1/4) = 0$. Therefore, if $g(l)$ is concave on $l \in (0, 1/2]$, then $\pi_1''(m) \leq 0$, $\forall m \in (0, 1/2]$. Then $m = 1/4$ is a global maximum and firm 1 has no incentive to deviate. It's easy to see that a necessary condition is $g''(l = 1/4) \leq 0$. Otherwise, $m = 1/4$ is a local minimum and firm 1 always has incentive to change p_1 slightly. ■

Proof of Proposition 2

We will divide the proof into two parts. Conditional on $f'(l)$ being concave, in part 1, we will prove that (i) $\pi^*(L = 1/2) \geq \pi^*(\tilde{L})$ holds, $\forall \tilde{L} \neq 1/2$. Then in part 2, we will prove that (ii) $\pi^*(L) \geq \pi^{mixed}(L)$ holds for any L.

Part 1: Show that $\pi^*(L=1/2) \geq \pi^*(\tilde{L})$, $\forall \tilde{L} \neq 1/2$.

From Lemma 1, we have

$$\pi^*(L) = \frac{1}{\frac{2}{f'(\frac{L}{2})} + \frac{2}{f'(\frac{1-L}{2})}} = \frac{1}{2} \frac{1}{\frac{1}{f'(\frac{L}{2})} + \frac{1}{f'(\frac{1-L}{2})}}.$$

It can be shown that if $f'(l)$ is concave in l, then $\frac{1}{f'(l)}$ is convex in l. Therefore

$$\frac{1}{f'(\theta x + (1-\theta)y)} \leq \theta \frac{1}{f'(x)} + (1-\theta)\frac{1}{f'(y)}, \forall \theta \in [0,1].$$

Setting $\theta = 1/2$, $x = L/2$ and $y = (1-L)/2$, we can obtain

$$\frac{1}{2}\frac{1}{f'(\frac{L}{2})} + \frac{1}{2}\frac{1}{f'(\frac{1-L}{2})} \geq \frac{1}{f'(\frac{1}{4})}.$$

Then

$$\pi^*(\tilde{L}) \leq \frac{1}{4f'(\frac{1}{4})} = \pi^*(L = 1/2).$$

Alternatively, it can be shown that $\frac{\partial \pi^*(L)}{\partial L} \geq 0$ $(L \in (0, 1/2])$ if and only if

$$f'(l)f'''(l) - 2\left(f''(l)\right)^2 < 0,$$

which is equivalent to $\frac{1}{f'(l)}$ being convex.

Part 2: Show that $\pi^*(L) \geq \pi^{mixed(L)}$, $\forall L$.

Consider a symmetric mixed strategy equilibrium where both firms choose prices according to a CDF $F(p)$ on $[\underline{p}, \bar{p}]$. Recall that $\tilde{p}$ denotes this mixed strategy which each firm plays. Next we prove that the upper bound of support of prices $\bar{p}$ can't be higher than p^*, which is given in Lemma 1.[15] Suppose not and $\bar{p} > p^*$. When choosing any price p_1 in the support, firm 1's expected profit is

$$E_{p_2}\pi_1(L, p_1, \tilde{p}) = (p_1 - c)\int_{\underline{p}}^{\bar{p}} [m(L, p_1, p_2) + n(L, p_1, p_2)]\, dF(p_2).$$

Taking derivative of $E_{p_2}\pi_1(L, p_1, \tilde{p})$ with respect to p_1, and evaluate it at $p_1 = \bar{p}$ (i.e., consider

[15] Note that $\underline{p}$, $\bar{p}$ and p^* are all functions of L. However, to ease on notation, we use $\underline{p}$, $\bar{p}$ and p^* instead.

how $E_{p_2}\pi_1(L, p_1, \tilde{p})$ changes when p_1 decreases slightly), we have

$$
\begin{aligned}
\frac{\partial E_{p_2}\pi_1(L, p_1, \tilde{p})}{\partial p_1}|_{p_1=\bar{p}} &= \int_{\underline{p}}^{\bar{p}} [m(L, \bar{p}, p_2) + n(L, \bar{p}, p_2)]\, dF(p_2)|_{p_1=\bar{p}} \\
&+ (p_1 - c)\int_{\underline{p}}^{\bar{p}} \left[\frac{\partial m(L, p_1, p_2)}{\partial p_1} + \frac{\partial n(L, p_1, p_2)}{\partial p_1}\right] dF(p_2)|_{p_1=\bar{p}} \\
&< \frac{1}{2} + (p_1 - c)\int_{\underline{p}}^{\bar{p}} \left[\frac{\partial m(L, p_1, p_2)}{\partial p_1} + \frac{\partial n(L, p_1, p_2)}{\partial p_1}\right] dF(p_2)|_{p_1=\bar{p}} \\
&= \frac{1}{2} - (\bar{p} - c)\int_{\underline{p}}^{\bar{p}} \left[\frac{1}{f'(m) + f'(L-m)} + \frac{1}{f'(n) + f'(1-L-n)}\right] dF(p_2)|_{p_1=\bar{p}}.
\end{aligned}
$$

The inequality is because with p_1 at the upper bound of the price support, firm 1's expected market share must be less than 1/2.

Since $f'(l)$ is concave in l, we have

$$\theta f'(x) + (1-\theta)f'(y) \leq f'(\theta x + (1-\theta)y), \forall \theta \in [0, 1].$$

By setting $\theta = 1/2$, $x = m$ and $y = L - m$, we have

$$\frac{1}{2}f'(m) + \frac{1}{2}f'(L-m) \leq f'\left(\frac{L}{2}\right) \Leftrightarrow f'(m) + f'(L-m) \leq 2f'\left(\frac{L}{2}\right)$$

$$\Leftrightarrow -\frac{1}{f'(m) + f'(L-m)} \leq -\frac{1}{2f'\left(\frac{L}{2}\right)}.$$

Similarly we can show that

$$-\frac{1}{f'(n) + f'(1-L-n)} \leq -\frac{1}{2f'\left(\frac{1-L}{2}\right)}.$$

Then

$$
\begin{aligned}
\frac{\partial E_{p_2}\pi_1(L, p_1, \tilde{p})}{\partial p_1}|_{p_1=\bar{p}} &< \frac{1}{2} - (\bar{p} - c)\int_{\underline{p}}^{\bar{p}} \left[\frac{1}{f'(m) + f'(L-m)} + \frac{1}{f'(n) + f'(1-L-n)}\right] dF(p_2)|_{p_1=\bar{p}} \\
&< \frac{1}{2} - (\bar{p} - c)\int_{\underline{p}}^{\bar{p}} \left[\frac{1}{2f'\left(\frac{L}{2}\right)} + \frac{1}{2f'\left(\frac{1-L}{2}\right)}\right] dF(p_2) \\
&= \frac{1}{2} - (\bar{p} - c)\left[\frac{1}{2f'\left(\frac{L}{2}\right)} + \frac{1}{2f'\left(\frac{1-L}{2}\right)}\right] \\
&< \frac{1}{2} - (p^* - c)\left[\frac{1}{2f'\left(\frac{L}{2}\right)} + \frac{1}{2f'\left(\frac{1-L}{2}\right)}\right] \\
&= \frac{1}{2} - \frac{1}{2} = 0.
\end{aligned}
$$

We have established that if $\bar{p} > p^*$, then $\frac{\partial E_{p_2}\pi_1(L,p_1,\tilde{p})}{\partial p_1}|_{p_1=\bar{p}} < 0$. That is, for $\epsilon \to 0^+$,

$$E_{p_2}\pi_1(L,\bar{p},\tilde{p}) < E_{p_2}\pi_1(L,\bar{p}-\epsilon,\tilde{p}).$$

Firm 1 would strictly prefer $p_1 = \bar{p} - \epsilon$ to $p_1 = \bar{p}$, contradicting the equilibrium setup that $\bar{p}$ is in the price support. Therefore $\bar{p} > p^*$ can't happen.

At $p_1 = \bar{p}$, firm 1's expected demand is less than 1/2. Moreover, $\bar{p} \le p^*$. It must be that

$$\pi^{mixed}(L) = E_{p_2}\pi_1(L,\bar{p},\tilde{p}) < \pi^*(L).$$

■

Proof of Lemma 2

Let $L \le 1/2$ denote firm 2's location, while firm 1 is located at 0. It is easy to see that welfare loss is minimized (thus social welfare is maximized) only when there is no price distortion. In this case, the loss of welfare is the following:

$$loss(L) = 2\int_{l=0}^{\frac{L}{2}} f(l)dl + 2\int_{l=0}^{\frac{1-L}{2}} f(l)dl.$$

Taking derivative with respect to L, we have

$$\frac{d\ loss(L)}{dL} = f\left(\frac{L}{2}\right) - f\left(\frac{1-L}{2}\right),$$

which is negative whenever $L < 1/2$ and equals 0 when $L = 1/2$. Therefore, $L = 1/2$ minimizes the loss of efficiency. ■

Proof of Lemma 3

We start with the case of linear transportation cost. Suppose that $f(l) = al$ with $a > 0$. We have explained that $L = 1/2$ is only one of the continuum of equilibrium location, among which firms are indifferent since all lead to the same equilibrium profit. Consumer surplus, however, depends on L. When profit is constant, any L that maximizes social surplus also maximizes consumer surplus. Since transportation cost is minimized when $L = 1/2$, consumer surplus must be maximized at $L = 1/2$ as well.

Suppose that the transportation cost function is quadratic, i.e., $f(l) = al^2$ with $a > 0$. It can be verified that $L = 1/2$ is the unique optimal choice of location, since the symmetric pure strategy equilibrium leads to the maximum profit when $L = 1/2$. Next we show that consumer surplus is also maximized when $L = 1/2$. With quadratic transportation cost, consumer surplus is the following,[16]

$$CS(L) = U - c - \frac{7}{8}aL + \frac{7}{8}aL^2 - \frac{a}{24}.$$

[16] It can be shown that if the transportation cost is linear-quadratic (e.g. $f(l) = al + bl^2$), then $\frac{d^2CS(L)}{dL^2}$ is independent of L, but it be either positive or negative, depending on the parameters a and b.

Consumer surplus is strictly concave in L, since

$$\frac{d^2CS(L)}{dL^2} = \frac{7}{4}a.$$

Then $L = 1/2$ is a global maximum. ■

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