Chapter 6. Dynamics

1 LDE: Scalar Equations

In this section, we look at single linear difference equations (LDE) in this section, and time t can take only integer values. Recall that $\frac{dy}{dt}$ is the derivative and dy is called the differential. In the discrete counterpart, we have $\frac{\Delta y}{\Delta t}$, and Δy is called the first difference.

We will define the first difference more specifically by adding a time subscript,

$$\Delta y_t \equiv y_{t+1} - y_t.$$

For example,

$$(1)\Delta y_t = 2; \quad (2)\Delta y_t = -0.2y_t.$$

Equations of these types are called difference equations. We can rewrite them as

$$(1)y_{t+1} - y_t = 2; \quad (2)y_{t+1} = 0.8y_t.$$

1.1 First-Order Difference Equation

A typical such equation is

$$y_{t+1} = ay_t + b \tag{1}$$

for some constants a, b. A solution of difference equation (1) is an expression for y_t in terms of the initial condition y_0, a, b and t.

The general solution will consist of two components: a *particular solution* y_p, which is *any* solution of the nonhomogeneous equation, and a *complementary function* y_c, which is the general solution of the homogenous equation (by setting $b = 0$).

We can easily see that the complementary function is $y_c = Aa^t$. Next we look for the particular solution. Pick a trial solution $y_t = k$, then the initial equation becomes

$$k = ak + b \Rightarrow k = \frac{b}{1-a},$$

assuming that $a \neq 1$.*

Then the general solution is

$$y_t = Aa^t + \frac{b}{1-a}.$$

Using initial condition,

$$y_0 = Aa^0 + \frac{b}{1-a} \Rightarrow A = y_0 - \frac{b}{1-a}.$$

There is also another method, where we transform the nonhomogeneous equation of variable y into a homogeneous equation of another variable. Specifically, we want to have something like

$$(y_{t+1} - c) = a(y_t - c) \Rightarrow z_{t+1} = az_t,$$

where the new variable $z_t = y_t - c$.

The general solution is

$$z_t = a^t z_0.$$

*If $a = 1$, then the solution is trivial.

We need to find out c,

$$(y_{t+1} - c) = a(y_t - c) \Rightarrow y_{t+1} - c = ay_t - ac$$

$$\Rightarrow y_{t+1} = ay_t + (1-a)c \Rightarrow \boxed{c = \frac{b}{1-a}}.$$

Plug it into the general solution, we can easily verify that this leads to the same solution from the previous method.

<u>Economic Application: The Basic Cobweb Model</u>

Consider the following dynamic supply and demand model

$$\text{Demand:} \quad Q_t^D = \alpha + aP_t, \quad a < 0.$$

$$\text{Supply:} \quad Q_t^S = \beta + bP_{t-1}, b > 0.$$

Setting demand equal to supply, we get the first-order linear difference equation

$$aP_t - bP_{t-1} = \beta - \alpha.$$

A particular solution to this equation is obtained by letting $P_t = P_{t-1} = \bar{P}$. Then,

$$\bar{P} = \frac{\beta - \alpha}{a - b}.$$

The general solution is

$$P_t = \bar{P} + k \left(\frac{b}{a}\right)^t,$$

where k is an arbitrary constant.

Since $b/a < 0$, the time path for P must exhibit oscillations. These oscillations will be convergent iff. $|b/a| < 1 \Leftrightarrow b < |a|$, i.e., the supply curve is flatter than the demand curve.

1.2 Second-Order Difference Equation

A simple variety of second-order difference equation takes the form

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c.$$

This equation is linear, second-order and nonhomogeneous, with constant coefficients. As in the first-order equation, the general solution consists of two components: a *general solution* to the homogeneous equation and a *particular solution* to the nonhomogeneous equation.

Particular Solution

We try $y_t = k$. Then

$$k + a_1 k + a_2 k = c \Rightarrow \boxed{k = \frac{c}{1+a_1+a_2}}.$$

General Solution

Our experience with the first-order difference equations taught us that the expression Aa^t plays a prominent role in the general solution. So we try a general solution $y_t = Aa^t$ and we need to determine the values of A and a. Substitute $y_t = Aa^t$, the initial equation becomes,

$$Aa^{t+2} + a_1 Aa^{t+1} + a_2 Aa^t = 0 \Rightarrow$$

$$Aa^t(a^2 + a_1 a + a_2) = 0.$$

The **auxiliary equation** associated with our difference equation is

$$x^2 + a_1 x + a_2 = 0.$$

There are two solutions: x_1 and x_2, according to the formula

$$x_1, x_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2},$$

both of which should appear in the general solution.

If x_1 and x_2 are distinct and real roots, then the general solution is

$$y_c = c_1 x_1^t + c_2 x_2^t.$$

If $x_1 = x_2 = r$ are equal real roots, then the general solution is

$$y_c = (c_1 + c_2 t) r^t.$$

If x_1, x_2 are complex conjugates

$$x_1, x_2 = h \pm vi,$$

then the general solution is

$$y_c = c_1 (h + vi)^t + c_2 (h - vi)^t.$$

Economic Application: Cobweb Model Again

In the basic model, the behavior implied for the farmer is unrealistic. Presumably farmers will not believe that the price of the output will not change, when the make decision in period $t-1$. A more realistic approach would have farmers basing their decisions on their expected prices, which we model as the following

$P_t^e = P_{t-1} - \rho(P_{t-1} - P_{t-2}) = (1-\rho)P_{t-1} + \rho P_{t-2}$, with $0 < \rho < 1$.

The new Cobweb model becomes

$$\text{Demand:} \quad Q_t^D = \alpha + aP_t, \quad a < 0.$$

$$\text{Supply:} \quad Q_t^S = \beta + bP_{t-1}^e, b > 0, 0 < \rho < 1.$$

Setting demand equal to supply and simplify, we can obtain,

$$P_t + c(1-\rho)P_{t-1} + c\rho P_{t-2} = \frac{\beta - \alpha}{\alpha},$$

where $c = -b/a > 0$.

The particular solution is found by setting $P_t = P_{t-1} - P_{t-2} = \bar{P}$ to obtain

$$\bar{P} = \frac{\beta - \alpha}{a - b}.$$

The general solution to the homogeneous equation depends on the roots of the auxiliary equation:

$$x^2 + c(1-\rho)x + c\rho = 0.$$

The roots are given by,

$$x_1, x_2 = \frac{-c(1-\rho) \pm \sqrt{c^2(1-\rho)^2 - 4c\rho}}{2}.$$

Convergence requires that the absolute value of both roots are less than one.

Assume that they are distinct and real, then the general solution is

$$P_t = \bar{P} + (c_1 x_1^t + c_2 x_2^t)$$

2 LDE: System of Equations

We looked at a single difference equation in the previous section. In this section, we will look at simultaneous difference equations, and we will rely on eigenvalues and eigenvectors.

2.1 Eigenvalues and Eigenvectors

Eigenvalues

Let A be a square matrix. An Eigenvalue of A is a number r which, when subtracted from each of the diagonal entries of A, converts A into a singular matrix.

Subtracting a scalar r from each diagonal entry of A is the same as subtracting r times the identity matrix I from A.

Therefore, r is an eigenvalue of A iff. $A - rI$ is a singular matrix.

Examples

$$A = \left(\begin{array}{ccc} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{array} \right), \quad B = \left(\begin{array}{cc} 2 & 0 \\ 0 & 3 \end{array} \right)$$

It can be easily seen that $r = 2$ is an eigenvalue of A, since subtracting 2 from each of its diagonal entries, the matrix becomes the following singular matrix.

$$\left(\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right).$$

For matrix B, its eigenvalues are 2 and 3 respectively.

Theorem 1 *(1) The diagonal entries of a diagonal matrix D are eigenvalues of D.*

(2) A square matrix A is singular iff. 0 *is an eigenvalue of A.*

For most matrices, one can't just look at the matrix to find out what number to subtract from its diagonal entries to make the matrix singular, and we have to go through the general method, i.e., solve the equation

$$\det(A - rI) = 0. \tag{2}$$

For an $n \times n$ matrix A, the left-hand side of equation (2) is an nth order polynomial in the variable r, called the characteristic polynomial of A.

The number r is an eigenvalue of A iff. r is a root of the characteristic polynomial of A.

An nth order polynomial has n roots (multiplicity and complex roots counted). Therefore, an $n \times n$ matrix has at most n eigenvalues.

Eigenvalues of Special Matrices

Let A and B be $n \times n$ matrices. Then A is similar to B if there exists a nonsingular matrix C such that

$$B = C^{-1}AC.$$

Theorem 2 *If A and B are similar matrices, they have the same eigenvalues.*

Proof.

$$|B-\lambda I| = |C^{-1}AC-\lambda C^{-1}C| = |C^{-1}(A-\lambda I)C|$$

$$= \frac{1}{|C|}|A-\lambda I||C| = |A-\lambda I|.$$

Therefore,

$$|B-\lambda I| = 0 \Leftrightarrow |A-\lambda I| = 0.$$

■

Theorem 3 *The eigenvalues of an idempotent matrix are either one or zero.*

Proof. Let A be idempotent and consider

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Premultiplying both sizes by A we have

$$A\mathbf{x} = A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x}.$$

Subtracting this equation from the previous one gives

$$\mathbf{0} = \lambda(\lambda - 1)\mathbf{x}.$$

Thus λ has to either zero or one since $\mathbf{x} \neq 0$.

■

Eigenvalues and $det(A)$, $r(A)$ and $tr(A)$

The trace of a square matrix is the sum of its diagonal entries.

Theorem 4 *Let A be a $k \times k$ matrix with eigenvalues $r_1, \cdots, r_k$. Then,*

(a) $r_1 + r_2 + \cdots + r_k =$ trace of A,

(b) $r_1 \cdot r_2 \cdots r_k =$ det(A), and

(c) the rank of A equals the number of nonzero eigenvalues of A.

Examples

The matrix $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ is singular, since its first row is twice the second row.

Therefore, 0 is an eigenvalue. By the previous theorem, the other eigenvalue is $2+2-0=4$.

Question

By the previous theorem, what do we know about the rank and trace of an idempotent matrix?

Eigenvectors

Recall from previous chapter that a square matrix B is singular iff. the system $B\mathbf{x}=\mathbf{0}$ has a nonzero solution.

Now suppose that r is an eigenvalue of the matrix A. Then $A-rI$ is singular matrix, implying that the system of equations $(A-rI)\mathbf{v}=\mathbf{0}$ has a nonzero solution $\mathbf{v}$.

This $\mathbf{v}$ is called an eigenvector of A corresponding to eigenvalue r.

Theorem 5 *Let A be an $n \times n$ matrix and r be a scalar. Then, the following statements are equivalent:*

(a) $A - rI$ is a singular matrix $\Leftrightarrow$ det$(A - rI) =$ 0.

(b) $(A - rI)\mathbf{v} = \mathbf{0}$ for some nonzero vector $\mathbf{v}$.

Example

Solve for the eigenvalues and eigenvectors for the following matrices.

$$A = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{pmatrix}$$

Solution

Start with matrix $A = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}$.

Suppose r is an eigenvalue of A, then

$$det(A - rI) = det \begin{pmatrix} -1 - r & 3 \\ -2 & 4 - r \end{pmatrix} = 0.$$

$$\Rightarrow (-1-r)\times(4-r)-(-2)\times 3 = 0 \Rightarrow r^2-3r+2 = 0$$

$$\Rightarrow (r - 1)(r - 2) = 0 \Rightarrow r_1 = 1, r_2 = 2.$$

Next we look for eigenvectors. First subtract $r_1 = 1$ from the diagonal entries of A and solve

$$(A - rI)\mathbf{v} = \begin{pmatrix} -2 & 3 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.$$

Note that $v_1 = 3$ and $v_2 = 2$ is a solution, and we say that one eigenvector is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

There are other eigenvectors, such as $(-3, -2)^T$, $(1, 2/3)^T$. In general, we choose the "simplest" of the nonzero candidates.

Now we use the other eigenvalue $r_2 = 2$, and the equation becomes

$$(A - rI)\mathbf{v} = \begin{pmatrix} -3 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0.$$

One solution is $v_1 = v_2 = 1$. The simplest eigenvector would be $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Next we solve for the eigenvalues and eigenvectors for matrix

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{pmatrix}$$

Its characteristics equation is,

$$det \begin{pmatrix} 1-r & 0 & 2 \\ 0 & 5-r & 0 \\ 3 & 0 & 2-r \end{pmatrix} = 0$$

$$\Rightarrow (1-r)(5-r)(2-r) - 2\times 3(5-r) = 0$$

$$\Rightarrow (5-r)[(r^2-3r+2)-6] = 0$$

$$\Rightarrow (5-r)(r-4)(r+1) = 0.$$

Therefore, the eigenvalues of B are $r = 5, 4, -1$.

Next we look for its eigenvectors. First use $r = 5$, we need to solve the equation

$$(B-5I)\mathbf{v} = \begin{pmatrix} -4 & 0 & 2 \\ 0 & 0 & 0 \\ 3 & 0 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= \begin{pmatrix} -4v_1 + 2v_3 \\ 0 \\ 3v_1 - 3v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

One solution is $v_1 = v_3 = 0, v_2 = 1$. Thus $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector for $r = 5$.

To find an eigenvector for $r = 4$, solve

$$(B - 5I)\mathbf{v} = \begin{pmatrix} -3 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= \begin{pmatrix} -3v_1 + 2v_3 \\ v_2 \\ 3v_1 - 2v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

One solution is $v_1 = 2, v_2 = 0$ and $v_3 = 3$.
Thus $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ is an eigenvector for $r = 4$.

Using the same method, we can show that
$\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is an eigenvector for $r = -1$.

2.2 Solving Systems of LDEs

Two-dimensional System: An Example

Now consider a system of two linear difference equations, in which the size of each variable depends linearly on the sizes of both variables in the previous period:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

If $b = c = 0$, then they are uncoupled and can be easily solved as two separate one-dimensional problems.

When they are coupled ($b \cdot c \neq 0$), the technique for solving the system is to find a change of variables that uncouples these equations.

General Two-dimensional Systems

Consider the following system of difference equations

$$z_{n+1} = Az_n$$

There is some matrix P such that $z = PZ$ or $Z = P^{-1}z$. Now write the original difference equation in the new variables Z:

$$\begin{aligned} Z_{n+1} &= P^{-1}z_{n+1}, \\ &= P^{-1}(Az_n) \\ &= (P^{-1}A)z_n \\ &= (P^{-1}A)(PZ_n) \\ &= (P^{-1}AP)Z_n \end{aligned}$$

Our goal is to choose the transformation P so that the transformed system $Z_{n+1} = (P^{-1}AP)Z_n$ is as simple as possible.

Specifically, if $(P^{-1}AP)$ is <u>diagonal</u>, then the transformed system of difference equations would be uncoupled and easily solved.

Let's compute, in the two-dimensional case, what kind of matrix P will lead to a diagonal $(P^{-1}AP)$.

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the two columns of the 2×2 matrix P, i.e., $P = (\mathbf{v}_1, \mathbf{v}_2)$, and write the diagonal matrix as $D = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$

Then

$$P^{-1}AP = D \Rightarrow AP = PD$$

$$\Rightarrow A(\mathbf{v}_1, \mathbf{v}_2) = (\mathbf{v}_1, \mathbf{v}_2) \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$$

$$\Rightarrow (A\mathbf{v}_1, A\mathbf{v}_2) = (r_1\mathbf{v}_1, r_2\mathbf{v}_2)$$

or simply $A\mathbf{v}_1 = r_1\mathbf{v}_1$ and $A\mathbf{v}_2 = r_2\mathbf{v}_2$.

This means that we want r_1 and r_2 to be eigenvalues of A, and $\mathbf{v}_1$ and $\mathbf{v}_2$ to be the corresponding eigenvectors.

Furthermore, this calculation shows that if P is a nonsingular matrix such that $P^{-1}AP$ is a diagonal matrix $\begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$, then r_1 and r_2 are eigenvalues of A and the columns of P are the corresponding eigenvectors.

<u>k-dimensional Systems</u>

Let $r_1, \cdots, r_k$ be eigenvalues of $k \times k$ matrix A. Let $\mathbf{v}_1, \cdots, \mathbf{v}_k$ be the corresponding eigenvectors. Form the matrix P whose jth column is eigenvector $\mathbf{v}_j$. Then

$$\begin{aligned} AP &= A \quad (\mathbf{v}_1, \cdots, \mathbf{v}_k) \\ &= (A\mathbf{v}_1, \cdots, A\mathbf{v}_k) \\ &= (r_1\mathbf{v}_1, \cdots, r_k\mathbf{v}_k) \\ &= (\mathbf{v}_1, \cdots, \mathbf{v}_k) \begin{pmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k \end{pmatrix} \\ &= P \begin{pmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k \end{pmatrix} \end{aligned}$$

Then

$$P^{-1}AP = \begin{pmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k \end{pmatrix}$$

Theorem 6 *Let A be a $k \times k$ matrix. Let $r_1, \cdots, r_k$ be its eigenvalues, and $\mathbf{v}_1, \cdots, \mathbf{v}_k$ be its eigenvectors. Form the matrix*

$$P = (\mathbf{v}_1, \cdots, \mathbf{v}_k)$$

whose columns are these k eigenvectors. If P is invertible, then

$$P^{-1}AP = \begin{pmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k \end{pmatrix} \tag{3}$$

Conversely, if $P^{-1}AP$ is a diagonal matrix D, the columns of P must be eigenvectors of A and the diagonal entries of D must be eigenvalues of A.

A key hypothesis of Theorem 6 is that the matrix P whose columns are eigenvectors of A is invertible, or the $k \times k$ matrix A has k linearly independent eigenvectors.

Theorem 7 *Let $r_1, \cdots, r_h$ be h distinct eigenvalues of the $k \times k$ matrix A. Let $\mathbf{v}_1, \cdots, \mathbf{v}_h$ be the corresponding eigenvectors. Then $\mathbf{v}_1, \cdots, \mathbf{v}_h$ are linearly independent.*

Next we apply Theorem 6 and 7 to solve a general linear system of equations: $\mathbf{z}_{n+1} = A\mathbf{z}_n$.

Suppose that the eigenvalues of A are real and distinct. We construct P as a matrix of eigenvectors of A.

The linear change of variables $\mathbf{z} = P\mathbf{Z}$ transforms the this system of difference equations to the uncoupled system:

$$\mathbf{Z}_{n+1} = \begin{pmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k \end{pmatrix} \mathbf{Z}_n.$$

We can easily solve this uncoupled system as

$$\begin{array}{rcl} (Z_1)_n & = & c_1 r_1^n \\ (Z_2)_n & = & c_2 r_2^n \\ \vdots & & \vdots \\ (Z_k)_n & = & c_k r_k^n \end{array}, \tag{4}$$

where $\mathbf{Z} = (Z_1, Z_2, \cdots, Z_k)^T$. Transforming equation (4) back to the $\mathbf{z}$-variables yields,

$$\mathbf{z}_n = \begin{pmatrix} (z_1)_n \\ \vdots \\ (z_k)_n \end{pmatrix} = P \begin{pmatrix} (Z_1)_n \\ \vdots \\ (Z_k)_n \end{pmatrix}$$

$$= (\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k) \begin{pmatrix} c_1 r_1^n \\ c_2 r_2^n \\ \vdots \\ c_k r_k^n \end{pmatrix}$$

$$= c_1 r_1^n \mathbf{v}_1 + c_2 r_2^n \mathbf{v}_2 + \cdots + c_k r_k^n \mathbf{v}_k.$$

We can then use the initial condition (by setting $n = 0$ in the above equation) to solve for $c_1, c_2, \cdots, c_k$.

Example

Consider the following general linear system of equations: $\mathbf{z}_{n+1} = A\mathbf{z}_n$, where

$$A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}.$$

It can be showed that the eigenvalues are $r_1 = 5$, $r_2 = -2$, and the corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Then the general solution is

$$\mathbf{z}_n = c_1 r_1^n \mathbf{v}_1 + c_2 r_2^n \mathbf{v}_n$$

$$= \begin{pmatrix} c_1 \times 5^n \times 1 + c_2 \times (-2)^n \times (-3) \\ c_1 \times 5^n \times 2 + c_2 \times (-2)^n \times 1 \end{pmatrix}.$$

Setting $n = 0$, we can obtain

$$\mathbf{z}_0 = \begin{pmatrix} c_1 - 3c_2 \\ 2c_1 + c_2 \end{pmatrix}.$$

With initial condition $\mathbf{z}_0$, e.g. $\mathbf{z}_0 = (-1, 5)^T$, we can obtain $c_1 = 2, c_2 = 1$. Then we can plug them back into the general solution,

$$\mathbf{z}_n = \begin{pmatrix} 2 \times 5^n - 3 \times (-2)^n \\ 4 \times 5^n + (-2)^n \end{pmatrix}.$$

<u>An alternative approach: The powers of a matrix</u>

Back to the system of linear equations

$$\mathbf{z}_{n+1} = A\mathbf{z}_n.$$

Start with an initial state $\mathbf{z}_0 \in \mathbf{R}^k$, we have

$$\mathbf{z}_n = A^n \mathbf{z}_0.$$

In general, there is no convenient formula for the entries of A^n in terms of the entries of A, unless A is a diagonal matrix,

$$D = \begin{pmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k \end{pmatrix},$$

in which case,

$$D^n = \begin{pmatrix} r_1^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k^n \end{pmatrix}.$$

However, if we find a nonsingular matrix P so that $P^{-1}AP$ is a diagonal matrix D, then

$$\begin{aligned} A &= PDP^{-1} \\ A^2 &= (PDP^{-1})(PDP^{-1}) \\ &= PD^2P^{-1} \\ \vdots & \quad \vdots \\ A^n &= PD^nP^{-1} \\ &= P \begin{pmatrix} r_1^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k^n \end{pmatrix} P^{-1}. \end{aligned}$$

Theorem 8 *Let A be a $k \times k$ matrix. Suppose that there is a nonsingular matrix P such that*

$$P^{-1}AP = \begin{pmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k \end{pmatrix}.$$

Then,

$$A^n = P \begin{pmatrix} r_1^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k^n \end{pmatrix} P^{-1}.$$

The solution of the corresponding system of difference equations $\mathbf{z}_{n+1} = A\mathbf{z}_n$ with initial vector $\mathbf{z}_0$ is

$$\mathbf{z}_n = P \begin{pmatrix} r_1^n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k^n \end{pmatrix} P^{-1}\mathbf{z}_0.$$

Example

Let's look at the previous example again: a linear system of equations $z_{n+1} = Az_n$, where

$$A = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}.$$

This time, we will use the power method to solve for the general solution.

We already know that

$$r_1 = 5, r_2 = -2, P = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}, D = \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix}.$$

(1) Verify that $P^{-1} = \begin{pmatrix} 1/7 & 3/7 \\ -2/7 & 1/7 \end{pmatrix}$.

(2) Verify that $A^n = PD^nP^{-1}$ for $n = 1, 2$.

(3) The general solution is

$$\mathbf{z}_n = A^n \mathbf{z}_0 = PD^nP^{-1}\mathbf{z}_0$$

$$= \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 5^n & 0 \\ 0 & (-2)^n \end{pmatrix} \begin{pmatrix} 1/7 & 3/7 \\ -2/7 & 1/7 \end{pmatrix} \mathbf{z}_0$$

$$= \frac{1}{7} \begin{pmatrix} 5^n & -3 \times (-2)^n \\ 2 \times 5^n & (-2)^n \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} \mathbf{z}_0$$

$$= \frac{1}{7} \begin{pmatrix} 5^n + 6 \times (-2)^n & 3 \times 5^n - 3 \times (-2)^n \\ 2 \times 5^n - 2 \times (-2)^n & 6 \times 5^n + (-2)^n \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \times 5^n - 3 \times (-2)^n \\ 4 \times 5^n + (-2)^n \end{pmatrix}.$$

It can be easily checked that this solution is the same as the solution using the previous method. However, this method is more computationally demanding.

2.3 Repeated Eigenvalues

Roots of Characteristic Polynomial

Recall that the eigenvalues of a $k \times k$ matrix A are simply the roots of the characteristic polynomial of A – the kth order polynomial

$$p(r) = \det(A - rI).$$

A kth order polynomial has k roots–counting multiple roots and complex roots.

There are three possibilities for the roots of $p(r)$:

(1) $p(r)$ has k distinct, real roots,

(2) $p(r)$ has some repeated roots, or

(3) $p(r)$ has some complex roots.

Examples

Distinct real roots

For matrix $\begin{pmatrix} -4 & 2 \\ -1 & -1 \end{pmatrix}$, the characteristic polynomial is

$$p(r) = (-4-r)\times(-1-r)-(-1)\times 2 = r^2+5r+6$$

$$= (r+2)(r+3) \Rightarrow r_1 = -2, r_2 = -3.$$

Repeated roots

For matrix $\begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$, the characteristic polynomial is

$$p(r) = (4-r)\times(2-r)-(-1)\times 1 = r^2-6r+9$$

$$= (r-3)^2 \Rightarrow r_1 = r_2 = 3.$$

Complex roots

For matrix $\begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix}$, the characteristic polynomial is

$$p(r) = (0-r)\times(2-r)-(-1)\times 2 = r^2-2r+2$$

$$= (r-1)^2+1 \Rightarrow r_1 = 1+i, r_2 = 1-i.$$

Previously we learned how to diagonalize a $k \times k$ matrix that has k distinct real eigenvalues. Next we will look at the case of repeated eigenvalues. We will focus on a simple case: 2×2 matrix.

2×2 Nondiagonalizable Matrices

We focus on 2×2 matrices in this section. Consider two matrices

$$A_1 = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}.$$

Both have only one distinct eigenvalue: $r = 3$. However, A_1 has only one independent eigenvector, while A_2 has two independent eigenvectors.

A matrix A which has an eigenvalue of multiplicity $m > 1$ but does not have m independent eigenvectors corresponding to this eigenvalue is called a <u>nondiagonalizable</u> matrix.

Theorem 9 *Let A be a 2×2 matrix with two equal eigenvalues. Then A is diagonalizable iff. A is already diagonal.*

<u>Question</u>

What do we do with a nondiagonalizable matrix like A_1 above?

<u>Solution</u>

Recall that we want $P^{-1}AP$ to be as simple as possible.

If we can't achieve a diagonal matrix

$$P^{-1}AP = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix},$$

we would want an "<u>almost diagonal</u>" matrix:

$$P^{-1}AP = \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix}.$$

Consider the system $\mathbf{z}_{n+1} = A\mathbf{z}_n$. The change of variables $\mathbf{z} = P\mathbf{Z}$ transforms the system to

$$\mathbf{Z}_{n+1} = \begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix},$$

or

$$\begin{aligned} X_{n+1} &= rX_n + Y_n \\ Y_{n+1} &= rY_n. \end{aligned}$$

The system is still coupled, but we can solve Y_n directly and then plug it into the first equation to solve for X_n.

Generalized Eigenvector

Let r be an eigenvalue of the matrix A. A (nonzero) vector $\mathbf{v}$ such that $(A - rI)\mathbf{v} \neq \mathbf{0}$ but $(A - rI)^m\mathbf{v} = \mathbf{0}$ for some integer $m > 1$ is called a generalized eigenvector for A corresponding to r.

Theorem 10 *Let A be a 2×2 matrix with two equal eigenvalues r. Then,*

(a) either A has two independent eigenvectors corresponding to r, in which case A is the diagonal matrix rI, or

(b) A has only one independent eigenvector, say $\mathbf{v}_1$. In this case, there is a generalized eigenvector $\mathbf{v}_2$ such that $(A - rI)\mathbf{v}_2 = \mathbf{v}_1$. If $P \equiv (\mathbf{v}_1, \mathbf{v}_2)$, then $P^{-1}AP = \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix}$. In this case, the general solution of the system of difference equations $\mathbf{z}_{n+1} = A\mathbf{z}_n$ is

$$\mathbf{z}_n = (c_0 r^n + n c_1 r^{n-1})\mathbf{v}_1 + c_1 r^n \mathbf{v}_2.$$

We will skip the case of (1) 3×3 matrices with repeated eigenvalues but not independent eigenvectors, and (2) matrices with complex eigenvalues.

2.4 Markov Processes

A stochastic process is a rule which gives the probability that the system will be in state i at time $n+1$, given the probabilities of its being in the various states in previous periods.

This probability could, in principle, depend on the whole previous history of the system, that is, on the states that existed at times $1, 2, \cdots, n$.

When the probability that the system is in any state i at time $n+1$ depends only on what state the system was in at time n, the stochastic process is called a Markov processes.

For a Markov processes, only the immediate past matters.

The key elements of a Markov process are

(1) the probability $x^i(n)$ that state i occurs at time period n;

(2) the <u>transition probabilities</u> m_{ij}, where m_{ij} is the probability that the process will be in state j at time $n+1$ if it is in state i at time n.

It is natural to put the transition probabilities into a matrix which we call a <u>transition matrix</u>:

$$M = \begin{pmatrix} m_{11} & \cdots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{k1} & \cdots & m_{kk} \end{pmatrix}.$$

Note that the probabilities are written so that the first (second) subscript indexes the current (next) period.

Therefore, the sum of the m_{ij}'s over j for each i, i.e, the sum of the elements of each row, must equal 1.

This definition of Markov Transition Matrix is a bit different from that in the book, where the first (second) subscript denotes the next (current) period. And the sum of elements of each column equals 1.

Sometimes such a matrix is denoted by $Q(x'|x)$ which can be understood this way: that Q is a matrix, x is the existing state, x' is a possible future state, and for any x and x' in the model, the probability of going to x' given that the existing state is x, is Q.

We assume that the probabilities m_{ij} are fixed and independent of n, and we say that the process is time homogeneous or that the transition probabilities are <u>stationary</u>.

Now the process can be written as

$$\begin{pmatrix} x^1(n+1) \\ \vdots \\ x^k(n+1) \end{pmatrix} = \begin{pmatrix} m_{11} & \cdots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{k1} & \cdots & m_{kk} \end{pmatrix}^T \begin{pmatrix} x^1(n) \\ \vdots \\ x^k(n) \end{pmatrix} \tag{5}$$

Let M be a Markov matrix. Then M is called a regular Markov matrix if M^r has only positive entries for some integer r.

If $r = 1$, i.e., M itself has only positive entries, M is called a positive matrix.

Theorem 11 *Let M be a regular Markov matrix. Then,*

(a) 1 is an eigenvalue of M of multiplicity 1;

(b) every other eigenvalue r of M satisfies $|r| < 1$;

(c) eigenvalue 1 has an eigenvector $\mathbf{w}_1$ with strictly positive components; and

(d) if we write $\mathbf{v}_1$ for $\mathbf{w}_1$ divided by the sum of its components, then $\mathbf{v}_1$ is a probability vector and each solution $\mathbf{x}_n$ of $\mathbf{x}_{n+1} = M\mathbf{x}_n$ tends to $\mathbf{v}_1$ as $n \to \infty$.

2.5 Symmetric Matrices

Theorem 12 *Let A be a $k \times k$ symmetric matrix. Then,*

(a) all k roots of the characteristic equation det$(A - rI) = 0$ are real numbers;

(b) eigenvectors corresponding to distinct eigenvalues are orthogonal; and

(c) even if A has multiple eigenvalues, there is a nonsingular matrix P whose columns $\mathbf{w}_1, \cdots, \mathbf{w}_k$ are eigenvectors of A such that

(i) $\mathbf{w}_1, \cdots, \mathbf{w}_k$ are mutually orthogonal to each other,

(ii) $P^{-1} = P^T$,

(iii) $$P^{-1}AP = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_k \end{pmatrix}.$$

A matrix P that satisfies the condition $P^{-1} = P^T$, or $P^T P = I$, is called an orthogonal matrix.

Example

$$B = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{pmatrix}$$

It can be showed that $r_1 = 2, r_2 = 3$ and $r_3 = 6$. The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Next we normalize the eigenvectors

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{u}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{u}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Then

$$P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{pmatrix}.$$

$$P^T P = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

$$= I_3.$$

Definiteness of symmetric matrices

In this section, we analyze the relationship between the definiteness of a quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ and the signs of eigenvalues of A.

Theorem 13 *Let A be a symmetric matrix. Then,*

(a) A is p.d. (p.s.d.) iff all eigenvalues of A are > 0 (≥ 0);

(b) A is n.d. (n.s.d.) iff all eigenvalues of A are < 0 (≤ 0);

(c) A is indefinite iff. A has a positive eigenvalue and a negative eigenvalue.

Proof. Let P be an orthogonal matrix so that

$$P^{-1}AP = P^T AP = \begin{pmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k \end{pmatrix}.$$

Let $\mathbf{x} \in \mathbf{R}_k$ be an arbitrary nonzero vector, and let $\mathbf{y} = P^{-1}\mathbf{x} = P^T\mathbf{x}$. Then, $\mathbf{y}$ is nonzero, and

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T P^T A P\mathbf{y}$$

$$= \mathbf{y}^T \begin{pmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_k \end{pmatrix} \mathbf{y}$$

$$= r_1 y_1^2 + \cdots r_k y_k^2,$$

where at least one of the y_i^2 is nonzero. The rest of the proof is straightforward. ■

3. ODE: Scalar Equations

Recall from the last chapter that the growth of funds in a savings account which pays interest annually at rate r satisfies the difference equation

$$y_{t+1} = (1 + r)y_t.$$

If the interest in the account is paid every Δt fraction of a year, then the equation changes slightly to

$$\frac{y_{t+\Delta t} - y_t}{y_t} = r \cdot \Delta t.$$

For example, if the interest is paid every month, then the monthly rate is $\frac{r}{12}$, and

$$\frac{y_{t+\frac{1}{12}} - y_t}{y_t} = r \cdot \frac{1}{12}.$$

Let $\Delta t \to 0$, we obtain the *differential equation*

$$\dot{y} = \frac{dy}{dt} = ry_t, \tag{6}$$

which states that the instantaneous percent rate of growth, $\dot{y}_t/y_t$, is a constant r.

In general, an ordinary differential equation is an expression which describes a relationship between a function of one variable and its derivative.

The solution to a differential equation is a function which satisfies that relationship.

Differential equations which describe a relationship between a function of several variables and its partial derivatives are called partial differential equations.

3.1 Definitions and Examples

An ordinary differential equation is an equation $\dot{y} = F(y,t)$.

If the expression $F(y,t)$ does not specifically involve t, we call it a time-independent or autonomous differential equation.

If the equation specifically involves t, we call the equation nonautonomous or time-dependent.

An equation which involves derivatives up to and including the ith derivative is called an ith order differential equation.

Examples

$F(y,t) = 2t, F(y,t) = y^2, F(y,t) = t^2 y, F(y,t) = t^2.$

3.2 Explicit Solutions

In this section, we will list the most important classes of differential equations that have explicit solutions and compute their solutions.

Linear first order equations

(1) $\dot{y} = ay$, with a being a constant. The general solution is

$$y(t) = ke^{at}.$$

Note that (i) $y(t) = ke^{at}$ is a solution, and (ii) $y(t) = ke^{at}$ is a general solution.

Example:

Let $\dot{y} = ry$ describes the amount of money in a bank account where interest is continuously compounded at annual rate r. The general solution is $y(t) = ke^{rt}$ – the bank account grows exponentially without bound. However, knowing r is not enough to determine the size of the account at any moment, since we don't know the size of the original deposit, or the initial value $k = y(0)$.

(2) $\dot{y} = ay + b$ and $a \neq 0$. The general solution is

$$y(t) = -\frac{b}{a} + ke^{at}.$$

The Dynamics behind Demand and Supply

Assume linear functions for demand and supply

$$Q^D = \alpha + aP$$

$$Q^S = \beta + bP$$

and the following dynamic adjustment process

$$\dot{P} = \frac{dP}{dt} = \lambda(Q^D - Q^S), \quad \lambda > 0.$$

Substituting for Q^D, Q^S gives

$$\dot{P} = \lambda(a-b)P + \lambda(\alpha - \beta).$$

Using the above formula, we get the general solution

$$P(t) = -\frac{\alpha - \beta}{a-b} + ke^{\lambda(a-b)t}.$$

Under what conditions would the price converge?

(3) $\dot{y} = a(t)y$. The general solution is

$$y(t) = ke^{\int^t a(s)ds}.$$

We use the fact that if

$$f(x,y) = \int_x^y g(t)dt,$$

then

$$\frac{\partial f}{\partial x} = -g(x), \frac{\partial f}{\partial y} = g(y).$$

(4) $\dot{y} = a(t)y + b(t)$. The general solution is

$$y(t) = [k + \int^t b(s)e^{-\int^s a(u)du}]e^{\int^t a(s)ds}.$$

Separable equations

A differential equation $\dot{y} = F(y,t)$ is called separable if $F(y,t)$ can be written as a product

$$F(y,t) = g(y) \cdot h(t)$$

for some functions g and h.

Examples:

Separable: $\dot{y} = y^2(t^2+t), \dot{y} = e^y e^t, \dot{y} = (y+1)/t$ and $\dot{y} = y^2 + 1$.

Not separable: $\dot{y} = y^2 + t^2, \dot{y} = a(t)y + b(t)$ and $\dot{y} = ty + t^2y^2$.

The solution of a separable equation $\dot{y} = g(y)h(t)$ involves a simple trick.

First, write the equation as

$$\frac{dy}{dt} = g(y)h(t)$$

Then move all the y-terms to one side of the equation, and all the t-terms to the other side:

$$\frac{dy}{g(y)} = h(t)dt,$$

and integrate the y-side w.r.t. y and the t-side w.r.t. t:

$$\int^y \frac{dy}{g(y)} = \int^t h(t)dt + c. \qquad (7)$$

If there are no initial conditions, try to write this solution as $y = y(t, c)$.

If there is an initial condition $y(t_0) = y_0$, drop the c in (7) and write y_0 as the lower limit of the integration on the left-hand side of (7) and t_0 as the lower limit of integration on the right-hand side of (7).

<u>Example:</u> $\dot{y} = t^2 y$

Write it as

$$\frac{dy}{dt} = t^2 y \Rightarrow \frac{dy}{y} = t^2 \Rightarrow \int^y \frac{dy}{y} = \int^t t^2 dt + c$$

$$\Rightarrow \ln y = \frac{t^3}{3} + c \Rightarrow y = e^{t^3/3+c} = ke^{t^3/3}.$$

<u>Initial value problem</u>

Consider the problem

$$\frac{dY}{dp} = e^{ap} \cdot e^{bY} \cdot d^c, \quad Y(q) = I,$$

where a, b, c, q, and I are positive constants.

Separating variables and including the initial conditions as the lower limits of the integration yields,

$$\int_I^Y e^{-bY} dY = \int_q^p e^c e^{ap} dp$$

This implies,

$$-\frac{1}{b}e^{-bY} + \frac{1}{b}e^{-bI} = \frac{e^c}{a}(e^{ap} - e^{aq}).$$

$\underline{\dot{y} = g(y)}$

This is a special case of the previous section with $h(t) = 1$, but we are mainly interested in explicitly integrating the $\int dy/g(y)$ term in equation (7).

Examples:

$\dot{y} = y^2, \dot{y} = y(a - by).$

The Solow-Swan Neoclassical Growth Model

Consider a production function for the economy

$$Y = Y(K, L).$$

Assume that the product function is well-behaved and exhibits CRTS. Then,

$$Y(K/L, 1) = \frac{Y}{L} \Rightarrow y = \phi(k),$$

where the lower case variables are the ratio of the corresponding capital variables to Labor L. We assume $\phi'(k) > 0, \phi''(k) < 0$.

Suppose that a constant proportion s of output is saved, $S = sY$. Then the consumption is $C = (1 - s)Y$.

Let δ denote the rate of depreciation of the capital stock. Then net investment is given by

$$K' = I - \delta K = S - \delta K = sY - \delta K \Rightarrow$$

$$\frac{K'}{L} = sy - \delta k = s\phi(k) - \delta k. \tag{8}$$

Now

$$K' = \frac{dK}{dt} = \frac{d}{dt}(kL) = k\frac{dL}{dt} + L\frac{dk}{dt}.$$

Assuming that the labor supply grows at rate

$$\frac{dL/dt}{L} = n,$$

then

$$K' = knL + Lk'.$$

Substitute this equation into equation (8) yields,

$$k' = s\phi(k) - (n + \delta)k.$$

This is a first-order nonlinear differential equation.

Steady state occurs when

$$k' = 0 \Rightarrow s\phi(k) = (n+\delta)k \text{ at } k^*.$$

Note that k^* is a function of s, the savings rate. The SS per capita consumption is

$$c^*(s) = \phi(k^*(s)) - (n+\delta)k^*(s),$$

which is maximized when

$$\phi'(k^*) = (n+\delta).$$

This is called the golden rule of capital accumulation.

Pure integration problems: $\dot{y} = h(t)$

It's straightforward if the function $h(t)$ can be explicitly integrated.

3.3 Linear Second Order Equations

Homogeneous Equations

Consider the general second order homogeneous equation in the following form

$$a\ddot{y} + b\dot{y} + cy = 0. \tag{9}$$

If $a = 0$, then (9) becomes a first order linear equation whose solution has the form $y = e^{rt}$.

To find the solutions when $a \neq 0$, it is natural to just plug $y = e^{rt}$ into (9), which becomes

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0 \Rightarrow (ar^2 + br + c)e^{rt} = 0$$

$$\Rightarrow ar^2 + br + c = 0.$$

One can use the quadratic formula to find the roots:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

There are three possibilities for two roots, depending on the sign of $b^2 - 4ac$:

(1) $b^2 - 4ac > 0$, two real unequal roots;

(2) $b^2 - 4ac = 0$, two equal roots;

(3) $b^2 - 4ac < 0$, two complex roots.

Theorem 14 *(1) If there are two unequal real roots ($r_1 \neq r_2$), the general solution of* (9) *is $y_t = k_1 e^{r_1 t} + k_2 e^{r_2 t}$;*

(2) if there are two equal real roots ($r_1 = r_2$), the general solution is $y_t = k_1 e^{r_1 t} + k_2 t e^{r_2 t}$;

(3) if the roots are complex $\alpha \pm i\beta$, then the general solution is $y_t = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$.

Nonhomogeneous Equations

Consider the following equation

$$a\ddot{y} + b\dot{y} + cy = g(t). \qquad (10)$$

Theorem 15 *Let* $y_p(t)$ *be any particular solution of the nonhomogeneous equation* (10)*. Let* $k_1y_1(t) + k_2y_2(t)$ *be a general solution of the corresponding homogeneous equation* $a\ddot{y} + b\dot{y} + cy = 0$*. Then, a general solution of* (10) *is*

$$y(t) = k_1y_1(t) + k_2y_2(t) + y_p(t)$$

3.4 Existence and uniqueness of solutions

We have discussed many types of differential equations that one can compute an explicitly solution. However, a solution may exist even if there is no closed form.

Theorem 16 *Fundamental Theorem of Differential Equations*

Consider the initial value problem

$$\dot{y} = f(y,t), \quad y(t_0) = y_0. \tag{11}$$

Suppose that f is a continuous function at the point (t_0, y_0). Then, there exists a $CD1$ function $y : I \to \mathbf{R}^1$ defined on an open interval $I = (t_0 - a, t_0 + a)$ about t_0 such that $y(t_0) = y_0$ and $\dot{y} = f(y,t)$ for all $t \in I$, that is, $y(t)$ is a solution of the initial value problem (11). *Furthermore, if f is $CD1$ at (t_0, y_0), then the solution $y(t)$ is unique; any two solutions of* (11) *must be equal to each other on the intersection of their domains.*

4. ODE: Systems of Equations

4.1 Introduction

Consider a simple model of a *predator-prey* system,

$$\begin{aligned} \frac{\dot{x}}{x} &= a - by \\ \frac{\dot{y}}{y} &= -c + dx \end{aligned} \quad \Rightarrow \quad \begin{aligned} \dot{x} &= x(a - by) \\ \dot{y} &= y(-c + dx) \end{aligned}$$

The general first order system of two differential equations can be written as

$$\begin{aligned} \dot{x} &= F(x, y, t) \\ \dot{y} &= G(x, y, t) \end{aligned} \tag{12}$$

A solution of (12) is a pair of functions of $t, x^*(t)$ and $y^*(t)$ such that for every t,

$$\dot{x}^*(t) = F(x^*(t), y^*(t), t)$$

$$\dot{y}^*(t) = G(x^*(t), y^*(t), t) \qquad (13)$$

System (12) is a first order system because it involves only the first derivative of both unknown functions.

If F and G do not depend on t, the system is called autonomous or time-independent.

We will only deal with autonomous systems in this section.

Remark 1 Every *second* order system of one variable can be naturally written as a *first* order system in two variables:

$$\begin{aligned} \dot{y} &= v \\ \dot{v} &= f(v, y, t). \end{aligned}$$

Remark 2 Every *nonautonomous* differential equation $\dot{y} = f(y, t)$ in y can be written as an *autonomous* system of two differential equations in (y, r):

$$\begin{aligned} \dot{y} &= f(y, r) \\ \dot{r} &= 1. \end{aligned}$$

This implies that one only need to work with autonomous systems.

<u>Existence and uniqueness</u>

The existence and uniqueness theorem for solution of differential equations of one variable, holds equally well for systems of differential equations.

If F and G are continuous function in a neighborhood of (x_0, y_0, t_0), then there exist functions $x^*(t)$ and $y^*(t)$ defined and continuous on an open interval $I = (t_0 - \epsilon, t_0 + \epsilon)$ about t_0 such that (13) holds for all $t \in I$, $x(t_0) = x_0$ and $y(t_0) = y_0$.

Furthermore, if F and G are $CD1$ functions, the solution of the initial value problem is unique.

4.2 Linear Systems via Eigenvalues

It is natural to begin our analysis of systems of differential equations with the study of linear systems. The general linear system of differential equations can be written as

$$\begin{array}{lllll} \dot{x}_1 & = a_{11}x_1 + & \cdots & + a_{1n}x_n \\ \vdots & \quad\vdots & & \quad\vdots \\ \dot{x}_n & = a_{n1}x_1 + & \cdots & + a_{nn}x_n \end{array} \tag{14}$$

or simply as $\dot{\mathbf{x}} = A\mathbf{x}$.

If A is a diagonal matrix, then (14) separates into n self-contained equations $\dot{x}_i = a_{ii}x_i$.

If some off-diagonal a_{ij}'s are not zero so that the equations are linked to each other, then we can use eigenvalues and eigenvectors of A to unlink system (14) into n more or less self-contained equations, just as we did with systems of linear difference equations.

Distinct real eigenvalues

Theorem 17 *Suppose that* $n \times n$ *matrix* A *has* n *distinct real eigenvalues* $r_1, \cdots, r_n$*, with the corresponding eigenvectors* $\mathbf{v}_1, \cdots, \mathbf{v}_n$*. Then, the general solution of the linear system* $\dot{\mathbf{x}} = A\mathbf{x}$ *of differential equations is*

$$\mathbf{x}_t = c_1 e^{r_1 t}\mathbf{v}_1 + c_2 e^{r_2 t}\mathbf{v}_2 + \cdots + c_n e^{r_n t}\mathbf{v}_n$$

Now let's walk through this step by step. Suppose that A has n distinct real eigenvalues $r_1, \cdots, r_n$, with the corresponding eigenvectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$:

$$A\mathbf{v}_i = r_i \mathbf{v}_i \tag{15}$$

Let P a the $n \times n$ matrix whose columns are these n eigenvectors:

$$P = (\mathbf{v}_1, \cdots, \mathbf{v}_n).$$

Then equation (15) can be written as

$$AP = PD, D \equiv \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{pmatrix}.$$

Since the eigenvalues are distinct, the eigenvectors are linearly independent, and thus P is nonsingular and invertible.

Now use the linear change of variables $\mathbf{y} = P^{-1}\mathbf{x}$. Then,

$$\begin{aligned} \dot{\mathbf{y}} &= P^{-1}\dot{\mathbf{x}} \\ &= P^{-1}A\mathbf{x} \\ &= P^{-1}AP\mathbf{y} \\ &= D\mathbf{y} \end{aligned}$$

Since D is diagonal, the system $\dot{\mathbf{y}} = D\mathbf{y}$ can be written as

$$\dot{y}_1 = r_1 y_1, \cdots, \dot{y}_n = r_n y_n.$$

Its solution is

$$\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{r_1 t} \\ \vdots \\ c_n e^{r_n t} \end{pmatrix}$$

Finally, we use the transformation $\mathbf{x} = P\mathbf{y}$ to return to the original coordinates $x_1, \cdots, x_n$:

$$\begin{aligned} \mathbf{x}(t) &= P\mathbf{y}(t) \qquad (16) \\ &= (\mathbf{v}_1, \cdots, \mathbf{v}_n) \begin{pmatrix} c_1 e^{r_1 t} \\ \vdots \\ c_n e^{r_n t} \end{pmatrix} \\ &= c_1 e^{r_1 t}\mathbf{v}_1 + c_2 e^{r_2 t}\mathbf{v}_2 + \cdots + c_n e^{r_n t}\mathbf{v}_n \end{aligned}$$

Note that Theorem 17 is still valid if A has multiple eigenvalues, as long as each eigenvector of multiplicity $h > 1$ has h linearly independent eigenvectors.

Complex eigenvalues

Theorem 18 *Let A be a real 2×2 matrix with complex eigenvalues $\alpha \pm i\beta$ and the corresponding eigenvectors $\mathbf{u} \pm i\mathbf{w}$. Then, the general solution of the linear system of differential equations $\dot{\mathbf{x}} = A\mathbf{x}$ is*

$$\mathbf{x}(t) = e^{\alpha t} \cos \beta t (C_1 \mathbf{u} - C_2 \mathbf{w}) - e^{\alpha t} \sin \beta t (C_2 \mathbf{u} + C_1 \mathbf{w}). \tag{17}$$

Multiple real eigenvalues

Theorem 19 *Suppose that the 2×2 matrix A has equal eigenvalues r and only one independent eigenvector $\mathbf{v}$. Let $\mathbf{w}$ be a generalized eigenvector for A. Then, the general solution of the linear system of differential equations $\dot{\mathbf{x}} = A\mathbf{x}$ is*

$$\mathbf{x}(t) = e^{rt}(c_1 \mathbf{v} + c_2 \mathbf{w}) + t e^{rt}(c_2 \mathbf{v}). \tag{18}$$

4.3 Solving Linear Systems by Substitution

In this section, we present how to solve linear systems of differential equations by substitution, instead of using eigenvalues and eigenvectors.

Consider the following two variable first order linear system:

$$\begin{aligned} \dot{y}_1 &= a_{11}y_1 + a_{12}y_2 \\ \dot{y}_2 &= a_{21}y_1 + a_{22}y_2 \end{aligned} \tag{19}$$

From the first equation, we can obtain

$$y_2 = \frac{1}{a_{12}}(\dot{y}_1 - a_{11}y_1) \Rightarrow \dot{y}_2 = \frac{1}{a_{12}}(\ddot{y}_1 - a_{11}\dot{y}_1).$$

Plug these into the 2nd equation of (19), and simplify, we get

$$\ddot{y}_1 - (a_{11} + a_{22})\dot{y}_1 + (a_{11}a_{22} - a_{12}a_{21})y_1 = 0 \tag{20}$$

4.4 Steady States and Their Stability

Write a typical first order of system of differential equations in $\mathbf{R}^n$ as

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, \cdots, y_n) \\ \vdots & \qquad \vdots \\ \dot{y}_n &= f_n(y_1, \cdots, y_n) \end{aligned} \tag{21}$$

or in vector notation $\dot{\mathbf{y}} = F(\mathbf{y})$, where $F \equiv (f_1, \cdots, f_n) : \mathbf{R}^n \to \mathbf{R}^n$.

We call a constant-function solution $y_1(t) = y_1^*, \cdots, y_n(t) = y_n^*$ of (21) a steady state.

A point $\mathbf{y}^* = (y_1^*, \cdots, y_n^*)$ is a steady state of system (21) iff.

$$f_i(y_1^*, \cdots, y_n^*) = 0, i = 1, \cdots, n;$$

in vector notation, $F(\mathbf{y}^*) = \mathbf{0}$.

Therefore, finding steady state solutions is simply a matter of solving n algebraic equations in n variables.

Let $\mathbf{y}^*$ be a steady state for the equations $\dot{\mathbf{y}} = F(\mathbf{y})$; that is $F(\mathbf{y}^*) = 0$.

We say that $\mathbf{y}^*$ is an <u>asymptotically stable steady state</u> if every solution $\mathbf{y}(t)$ which starts near $\mathbf{y}^*$ converges to $\mathbf{y}^*$ as $t \to \infty$.

A steady state solution $\mathbf{y}^*$ of the system $\dot{\mathbf{y}} = F(\mathbf{y})$ is called <u>globally asymptotically stable</u> if just about <u>every</u> solution of $\dot{\mathbf{y}} = F(\mathbf{y})$ tends to $\mathbf{y}^*$ as $t \to \infty$.

More precisely, steady state $\mathbf{y}^*$ is globally asymptotically stable if for any $\mathbf{y}_0$, the solution of the initial value problem $\dot{\mathbf{y}} = F(\mathbf{y})$, $\mathbf{y}(0) = \mathbf{y}_0$ tends to $\mathbf{y}^*$ as $t \to \infty$.

A steady state $\mathbf{y}^*$ of the system $\dot{\mathbf{y}} = F(\mathbf{y})$ is called <u>neutrally stable</u> if it is not locally asymptotically stable and if all solutions which start close enough to $\mathbf{y}^*$ stay close to $\mathbf{y}^*$ as $t \to \infty$.

If a steady state $\mathbf{y}^*$ of $\dot{\mathbf{y}} = F(\mathbf{y})$ is asymptotically stable or neutrally stable, we call it <u>stable</u>. If it is neither asymptotically stable nor neutrally stable, we call it <u>unstable</u>.

<u>Stability of linear systems via eigenvalues</u>

To study the stability of the origin as an equilibrium of linear systems in n dimensions, it is most convenient to use formulas (16), (17) and (18) for the general solutions of $\dot{\mathbf{y}} = A\mathbf{x}$ to determine directly the stability of the steady state at $\mathbf{0}$.

In general, these solutions are a sum of terms of the form

$$\text{constant } \cdot e^{(\text{eigenvalue})\cdot t} \cdot \text{ eigenvector},$$

at least for real eigenvalues. Therefore, as $t \to \infty$,

$$\mathbf{x}(t) \to \mathbf{0} \iff \text{each } e^{r_i t} \to 0 \iff \text{each } r_i < 0.$$

If one $r_i > 0$, then $e^{r_i t}$ goes to infinity and drags the other terms of solutions with it.

In the case of complex roots, the $\cos\beta t-$ and $\sin\beta t-$ components of solution (17) oscillate about 0; the stability of $\mathbf{0}$ is determined by the $e^{\alpha t}$ factor.

If $\alpha < 0$, each $\mathbf{x}(t) \to \mathbf{0}$, and $\mathbf{0}$ is asymptotically stable.

If $\alpha > 0$, every $\mathbf{x}(t)$ is unbounded, and $\mathbf{0}$ is unstable.

If $\alpha = 0$, $\mathbf{x}(t)$ oscillates about $\mathbf{0}$ and $\mathbf{0}$ is neutrally stable.

Finally we need to consider solution (18) when A has repeated eigenvalues.

If $r > 0$, (18) goes to infinity as $t \to \infty$.

If $r < 0$, $e^{rt} \to 0$ and expression (18) goes to $\mathbf{0}$.

If $r = 0$, (18) becomes $c_1\mathbf{v} + c_2(t\mathbf{v} + \mathbf{w})$, which tends to infinity if $c_2 \neq 0$.

Stability of Nonlinear Systems

We turn now to the development of calculus criteria for the stability of a steady state of a nonlinear system of autonomous differential equations.

Theorem 20 *(1-dimension) Let y_0 be a steady state of the $CD1$ differential equation $\dot{y} = f(y)$ on the line; so $f(y_0) = 0$. If $f'(y_0) < 0$, then y_0 is asymptotically stable. If $f'(y_0) > 0$, then y_0 is unstable.*

See Figure 24.13 on page 668.

In a system $\dot{\mathbf{x}} = F(\mathbf{x})$ on $\mathbf{R}^n$ with steady state y^*, the Jacobian matrix $DF(\mathbf{y}^*)$ replaces the derivative of $f'(y^*)$.

The following theorem is the natural analogue of the n-dimensional linear results and the one-dimensional nonlinear results.

Theorem 21 *(n-dimension)* *Let $\mathbf{y}^*$ be a steady state of the first order system of differential equations $\dot{\mathbf{y}} = F(\mathbf{y})$ on $\mathbf{R}^n$, where F is a $CD1$ function from $\mathbf{R}^n$ to $\mathbf{R}^n$.*

(a) If each eigenvalue of the Jacobian matrix $DF(\mathbf{y}^*)$ of F at $\mathbf{y}^*$ is negative or has negative real part, then $\mathbf{y}^*$ is an asymptotically stable steady state of $\dot{\mathbf{y}} = F(\mathbf{y})$.

(b) If $DF(\mathbf{y}^*)$ has at least one positive real eigenvalue or one complex eigenvalue with positive real part, then $\mathbf{y}^*$ is an unstable steady state of $\dot{\mathbf{y}} = F(\mathbf{y})$.

5 Dynamic Optimization

In previous chapters we looked at *static* optimization problems. In this chapter we want to look at dynamic optimization problems, optimization problems that take place over time.

When time is treated as a continuous variable, dynamic optimization involves maximizing or minimizing the value of an integral. Originally such problem belongs to a branch of mathematics called the **calculus of variations**. Later with the work of Pontryagin, the theory was developed to what is now known as **optimal control theory**. At the same time, Bellman developed **dynamic programming**, a different method that is particularly suited to the case where time is treated as a discrete variable.

5.1 Continuous Time

Dynamic Optimization vs. Static Optimization

Consider the problem of optimal management of a fishery. Suppose a fishing operator has the rights to harvest fish from a given lake from time $t = 0$ to some terminal date $t = T$. Let $x(t)$ be the stock of fish in the lake and $u(t)$ the amount of fish harvested at time t. All fish caught can be sold at a fixed price p. The cost of fishing depends positive on $u(t)$ and negatively on $x(t)$. The cash flow for the operator at time t is

$$pu(t) - C(u(t), x(t)).$$

The value of the fishing rights is the present value of the stream of cash flow,

$$V = \int_0^T [pu(t) - C(u(t), x(t))]e^{-rt}dt.$$

The operator wishes to maximize V but he/she is constrained by the effect of the harvest has on the stock. Suppose that the growth rate of the fish stock is given by,

$$x'(t) = f(x(t)) - u(t).$$

The initial stock of fish in the lake is $x(0) = x_0$. Suppose that the license requires that a stock of fish $x(T) = x_T$ must be present in the lake when operation ceases.

The problem facing the fishing operator is

$$\max V = \int_0^T [pu(t) - C(u(t), x(t))]e^{-rt} dt$$

subject to

$$x'(t) = f(x(t)) - u(t).$$

and the conditions

$$x(0) = x_0, x(T) = x_T.$$

This example typifies an optimal control problem. It differs from the static optimization problem in the following was:

- Optimization takes place over a planning horizon, from $t = 0$ to $t = T$.

- The integrand that is maximized is a functional rather than a function, that is, it's a function of functions.

- It contains two types of variables: $x(t)$ is the state variable and $u(t)$ is the control variable.

- One of the constraints is a differential equation, which tells us how the choice of the control variable affects the state variable. This constraint is called the state equation.

- It contains initial and terminal value conditions.

5.2 Solving the Continuous Time Problem

Consider the following problem of optimal control:

$$\max \int_0^T F(t, y, u)dt$$

subject to

$$y'(t) = f(t, y, u).$$

and the conditions

$$y(0) = y_0, y(T) \text{ free.}$$

<u>Pontryagin's Maximum Principle</u>

The key to the optimal control theory is a first-order necessary condition known as the *maximum principle*. The statement of the maximum principle involves an approach that is akin to the Lagrangian function and the Lagrangian multiplier variable. For optimal control problems, these are know as the <u>Hamiltonian function</u> and <u>costate variable</u>.

Recall the three variables: time t, state variable y and control variable u. We now introduce a new variable known as the costate varialbe and denoted by $\lambda(t)$. Like the Lagrangian multiplier, the costate variables measures the shadow price of the state variable.

The Hamiltonian (function) is defined as

$$H(t, y, u, \lambda) \equiv F(t, y, u) + \lambda(t) f(t, y, u).$$

The maximum principle – a first-order necessary condition, requires us to choose u so as to maximize the Hamiltonian h at *every point of time*.

Suppose that H differentiable and yields an interior solution. Then the principle requires:

(i) $\frac{\partial H}{\partial u} = 0$

(ii) $y' = \frac{\partial H}{\partial \lambda}$ (equation of motion for the state variable or state equation)

(iii) $\lambda' = -\frac{\partial H}{\partial y}$ (equation of motion for the costate variable or costate equation)

(iv) $\lambda(T) = 0$ (transversality condition)

The state equation and costate equation constitute a system of differential equations. Thus we need two boundary conditions to determine the two arbitrary constants that will arise in the process of solution. If both the initial state $y(0)$ and the terminal state $y(T)$ are fixed, then these specification can be used to determine the constants. However, if the terminal state is not fixed, then something called a *transversality condition* must be included, to fill the gap left by the missing boundary condition.

Example

Find the optimal control path that will

$$\max \int_0^1 (y - u^2)dt$$

subject to $y' = 5$

and $y(0) = 5 \quad y(1)$ free.

The Hamiltonian for this problem is,

$$H = y - u^2 + \lambda u.$$

Applying the FOC,

$$\frac{\partial H}{\partial u} = -2u + \lambda = 0 \Rightarrow u(t) = \frac{\lambda}{2} \Rightarrow y' = \frac{\lambda}{2}.$$

The costate equation is

$$\lambda' = -\frac{\partial H}{\partial y} = -1.$$

The last two equations constitute the differential equation system for this problem. We solve λ first,

$$\lambda(t) = c_1 - t.$$

By transversality condition,

$$\lambda(1) = 0 \Rightarrow c_1 = 1 \Rightarrow \lambda^*(t) = 1 - t.$$

Then

$$y' = \frac{1-t}{2} \Rightarrow y(t) = \frac{1}{2}t - \frac{1}{4}t^2 + c_2.$$

Using the initial condition

$$y(0) = 5 \Rightarrow c_2 = 5.$$

Then the optimal control path is

$$u^*(t) = \frac{1-t}{2},$$

and the optimal path of the state variable is

$$y^*(t) = \frac{1}{2}t - \frac{1}{4}t^2 + 5.$$

Economic Application

Consider the following lifetime utility maximization problem

$$\max \int_0^T U(C(t))e^{-\delta t}dt$$

subject to $K' = rK(t) - C(t)$

and $K(0) = K_0 \quad K(T) \geq 0.$

5.3 Discrete Time

Consider the following infinite horizon dynamic programming problem,

$$\sum_{t=0}^{\infty} \beta^t f(x_t, u_t),$$

subject to

$$x_{t+1} = g(x_t, u_t)$$

and the initial condition x_0.

The solution takes the form of a function h mapping state x_t into control u_t

$$u_t = h(x_t)$$

$$x_{t+1} = g(x_t, u_t),$$

where iterating on the two equations above solves the maximization problem.

We want to jointly find V and h that satisfy the *Bellman Equation:*

$$V(x) = \max_u \{f(x, u) + \beta V[g(x, u)]\}.$$