# Chapter 7. Other Topics

## 1. Integration[*]

### 1.1 Basic Integration

A function on a bounded interval $I = [a, b]$ is piecewise continuous if it is continuous everywhere except on a finite number of points in $I$ and that at every point where it is not continuous it admits left and right limits.

Riemann Integration

For piecewise continuous functions on interval $I$, we define for any $a, b \in I$, $a < b$:

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[*] This section is mostly based on Markus M. Mobius's Math Econ notes. More information can be found at http://www.courses.fas.harvard.edu/∼ec2030/.

$$\int_a^b f(x)dx = \lim_{n \to +\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(a + \frac{k}{n}(b-a))$$

By definition, $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

The integral can be usefully interpreted as an area. One standard application of the integral is to find consumer surplus if a demand curve is given.

**Proposition 1** *Useful properties of the integral:*

- $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

- $\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$

- $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

We can define the following function $F(x)$:

$$F(x) = \int_a^x f(s)ds$$

**Theorem 1** *Fundamental Theorem of Calculus*

*The function $F(x)$ is differentiable if $f(\cdot)$ is continuous. Then we have $F'(x) = f(x)$.*

Sketch of the proof

$$F' = \frac{F(x+h) - F(x)}{h} = \frac{\int_x^{x+h} f(x)dx}{h}$$

It is easy to see in a picture, that the area of this small integral is approximately $f(x) \times h$.

From this theorem, we can see that integration is the "opposite" of differentiation. We call a function $F(x)$ a primitive of $F'(x)$.

**Proposition 2** *Assume that* $f$ *is continuous over the interval* $[a, b]$ *and that there exists functions* $F$ *and* $G$ *such that* $F'(x) = G'(x) = f(x)$. *Then* $F$ *and* $G$ *differ only by a constant, i.e.* $F(x) = G(x) + c$.

**Theorem 2** *Assume* $G$ *is a primitive of* $f$ *over an interval* $[a, b]$. *Then the following holds:*

$$\int_a^b f(x)dx = G(b) - G(a).$$

## 1.2 Useful Integrals and Tricks

- $\int_a^b \frac{1}{x}dx = \ln(b) - \ln(a)$

- $\int_a^b e^x dx = e^b - e^a$

- $\int_a^b x^\alpha dx = \frac{b^{\alpha+1} - a^{\alpha+1}}{\alpha+1}$

**Theorem 3** *Change of variables*

*Let $J_1$ and $J_2$ be intervals (with more than one point): Let $f : J_1 \to J_2$ and $g : J2 \to R$ be continuous. Assume that $f$ is differentiable and $f'$ is continuous. Then $\forall a, b \in J_1$,*

$$\int_a^b g(f(x))f'(x) = \int_{f(a)}^{f(b)} g(u)du.$$

Example

Consider the integral $I = \int_0^1 2xe^{x^2}dx$. Say $u = e^{x^2}$ and $du = 2xe^{x^2}dx$. Then we can write $I = \int_1^e du = e - 1$.

**Theorem 4** *Integration by part*

*Suppose $F$ and $G$ are differentiable on $[a, b]$. Suppose $F' = f$ and $G' = g$ are continuous. Then:*

$$\int_a^b f(t)G(t)dt = [F(b)G(b) - F(a)G(a)]$$

$$-\int_a^b F(t)g(t)dt.$$

You can derive this formula quickly by noticing that $[FG]' = fG + Fg$.

**Theorem 5** *Taylor expansion*

*Let $f : I \to R$ be a function of class $C^{m+1}$ (continuous $m+1$th derivative) and $x \in I$. We have*

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \ldots +$$

$$\frac{f^{(n)}(x)}{n!}h^n + o(h^n).$$

*with $o(h^n)/h^n \to 0$.*

If the function $f$ has more than one variable, then

$$f(x+h) = f(x) + \sum_i \frac{\partial f}{\partial x_i} h_i + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} h_i h_j$$

$$+\epsilon(h) \sum_j h_j^2$$

Some common expansions:

– $e^x = 1 + x + \ldots + x^n/n! + o(x^n)$

– $\ln(1+x) = x - x^2/2 + \ldots + (-1)^{(n+1)} x^n/n + \ldots$

– $(1+x)^\alpha = 1 + \alpha x + \alpha(\alpha-1)x^2 + \ldots + \alpha \ldots (\alpha - n + 1) x^n/n! + \ldots$

# 2. Convexity

Convex set

A subset $S$ of vector space $V$ is convex iff $\forall x, y \in S$, and $\forall \lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda)y \in S$.

Some comments:

– $\lambda x + (1 - \lambda)y$ is on the segment joining points $x$ and $y$.

– Convex sets can be flat (segment in $\mathbf{R}^2$)

– In $\mathbf{R}$ a set is convex iff it is an interval.

## 2.1 Convex and Concave Functions of One Variable

**Proposition 3** *Assume that* $f$ *is concave. For fixe* $a$*, the function* $\frac{f(x)-f(a)}{x-a}$ *decreases* *with $x$. The statement is reversed for convex function.*

**Proposition 4** *Let $I$ be an open interval in $R$ and $f : I \to R$ be a real function on $I$. Let us assume that $f$ is differentiable on $I$.*

*– $f$ is concave iff $f(y) - f(x) \leq f'(x)(y - x)$ for any $x, y \in I$.*

*– $f$ is convex iff $f(y) - f(x) \geq f'(x)(y - x)$ for any $x, y \in I$.*

**Proposition 5** *If $f$ is a function defined on a convex set $V \subset R^m$. Assume that $f$ is differentiable on $V$. Then*

*– $f$ is concave iff $f(\mathbf{y}) - f(\mathbf{x}) \leq f'(\mathbf{x})(\mathbf{y} - \mathbf{x}) = \sum_i \partial_i f(\mathbf{x})(y_i - x_i)$ for any $\mathbf{x}, \mathbf{y} \in I$.*

*– $f$ is convex iff $f(\mathbf{y}) - f(\mathbf{x}) \geq f'(\mathbf{x})(\mathbf{y} - \mathbf{x}) = \sum_i \partial_i f(x)(y_i - x_i)$ for any $\mathbf{x}, \mathbf{y} \in I$.*

## 2.2 Risk Aversion and Jensen Inequality

**Lemma 1** *Let $I$ be an interval, take $u : I \to R$, concave and an n-tuple $(p_1, \cdots, p_n)$ of positive numbers such that:*

$$p_1 + \ldots + p_n = 1.$$

*Then $\forall x_1, \cdots, x_n \in I$:*

$$u(p_1 x_1 + \cdots + p_n x_n) \geq p_1 u(x_1) + \cdots + p_n u(x_n).$$

This can be proved by induction on $n$.

**Theorem 6** *Jensen's Inequality*

*If $u$ concave and $f$ is a continuous function on $I$ then the following holds:*

$$u\left(\int_I x f(x) dx\right) \geq \int_I u(x) f(x) dx.$$

Proof Simply use the definition of integration and apply the above lemma to every single finite sum.

We can represent preferences over lotteries with a von-Neumann-Morgenstern (vMN) expected utility function. The utility of a lottery represented by a density function $f$ over the real line has the expected value:

$$u(f) = \int_{-\infty}^{\infty} u(x) f(x) dx$$

If the lottery is discrete with probabilities $p_1, \ldots, p_n$, then

$$u(f) = \sum_i p_i u(x_i).$$

The interpretation of Jensen's inequality is that if the vNM utility function is concave, then the decision maker prefers the expected outcome over an uncertain lottery – we say that she is risk-averse.

# 3. Properties of Continuous or Differentiable Functions

**Theorem 7** *(Min-Max Theorem)* *Let $f$ be a continuous function on a closed interval $[a, b]$. Then there is some $c \in [a, b]$ where $f$ reaches its maximum and some $d \in [a, b]$ where $f$ reaches its minimum.*

**Theorem 8** *(Weierstrass's Theorem)* *Let $F : C \to \mathbf{R}^1$ be a continuous function whose domain is a compact subsect $C$ in $\mathbf{R}^n$. Then, there exist points $\mathbf{x}_m$ and $\mathbf{x}_M$ such that $F(\mathbf{x}_m) \leq F(\mathbf{x}) \leq \mathbf{x}_M$ for all $\mathbf{x} \in C$.*

**Theorem 9** *(Intermediate Value Theorem)* *Let $f$ be a continuous function on a closed interval $[a, b]$. Then for any $m$ in the interval $[f(a), f(b)]$ or $([f(b), f(a)])$, there is some $c \in [a, b]$ such that $f(c) = m$.*

**Theorem 10** *(Rolle's Theorem) Suppose that* $f : [a, b] \to \mathbf{R}^1$ *is continuous on* $[a, b]$, *and* $CD1$ *on (a,b). If* $f(a) = f(b) = 0$, *then there is a point* $c \in (a, b)$ *such that* $f'(c) = 0$.

**Theorem 11** *(Mean value Theorem) Let* $f : U \to \mathbf{R}^1$ *be a* $CD1$ *function on a (connected) interval* $U$ *in* $\mathbf{R}^1$. *For any points* $a, b \in U$, *there is a point* $c$ *between* $a$ *and* $b$ *so that*

$$f(b) - f(a) = f'(c)(b - a).$$

**Theorem 12** *Let* $F : U \to \mathbf{R}^1$ *be a* $CD1$ *function defined on an open set* $U$ *in* $\mathbf{R}^n$. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two points in* $U$ *such that the line segment* $l(\mathbf{a}, \mathbf{b})$ *from* $\mathbf{a}$ *to* $\mathbf{b}$ *lies in* $U$. *Then, there is a point* $\mathbf{c} \in l(\mathbf{a}, \mathbf{b})$ *such that*

$$F(\mathbf{b}) - F(\mathbf{a}) = DF(\mathbf{c})(\mathbf{b} - \mathbf{a}).$$