Solution to problem Set 1

ECON 5153

1. i) $y = x^3 + 2x^2 - 4x$. First take derivatives

$$\frac{dy}{dx} = 3x^2 + 4x - 4 = (3x-2)(x+2), \text{ and } \frac{d^2y}{dx^2} = 6x + 4.$$

This function is CD2. Set $\frac{dy}{dx} = 0$ and solve for x, we obtain $x_1 = 2/3, x_2 = -2$. These are the only critical points.

When $x \in (-2, 2/3)$, $\frac{dy}{dx} < 0$, implying that the function is decreasing in x when $x \in (-2, 2/3)$. Similarly one can show that the function is increasing in x when $x \in (-\infty, -2) \cup (2/3, \infty)$.

At $x = 2/3$, $\frac{d^2y}{dx^2} > 0$, thus $x = 2/3$ a local minimum and $f(x = 2/3) = -\frac{40}{27}$. At $x = -2$, $\frac{d^2y}{dx^2} < 0$, thus $x = -2$ is a local maximum and $f(x = -2) = 8$.

The function has no global maximum/minimum, since $y \to \infty$ when $x \to \infty$ and $y \to -\infty$ when $x \to -\infty$.

ii) $y = |x|$. This function is differentiable everywhere except $x = 0$, which is only the critical point. It can be shown that it's the only local and global minimum and the corresponding $y = 0$. When $x > 0$, $y = x$ and $\frac{dy}{dx} = 1$, thus the function is increasing. When $x < 0$, $y = -x$ and $\frac{dy}{dx} = -1$, thus the function is decreasing.

There is no local or global maximum, as $y \to \infty$ when $x \to -\infty$ or ∞.

iii) $y = \frac{1}{2x+1}$. First note that the function is not defined at $x = -1/2$. When $x \neq -1/2$, $\frac{dy}{dx} = -\frac{2}{(2x+1)^2} < 0$. So the function is always decreasing in its domain. $\frac{d^2y}{dx^2} = \frac{8}{(2x+1)^3}$. It is positive (thus the function is convex) when $x > -1/2$, and negative (the function is concave) when $x < -1/2$. This function does not have a critical point, therefore, there is no local minimum/maximum. Moreover, since this function has no boundaries, there is no global minimum/maximum either.

2. i) $y = \frac{1}{\sqrt{x^2-2x-3}}$. We need $x^2 - 2x - 3 > 0$.

$$x^2 - 2x - 3 = (x+1)(x-3) > 0 \Leftrightarrow x < -1 \text{ or } x > 3.$$

Then the domain is $D = (-\infty, -1) \bigcup (3, \infty)$. Note that $\sqrt{x^2 - 2x - 3}$ can take any positive values, thus the range is $S = (0, \infty)$.

ii) $\frac{\ln(x)}{x^3-1}$. First we need $x > 0$ to have $\ln(x)$ defined. Then we need $x^3 - 1 \neq 0$ since it is in the denominator. This implies $x \neq 1$. The domain is $D = (0, 1) \bigcup (1, \infty)$. When $x < 1$, both $\ln(x)$

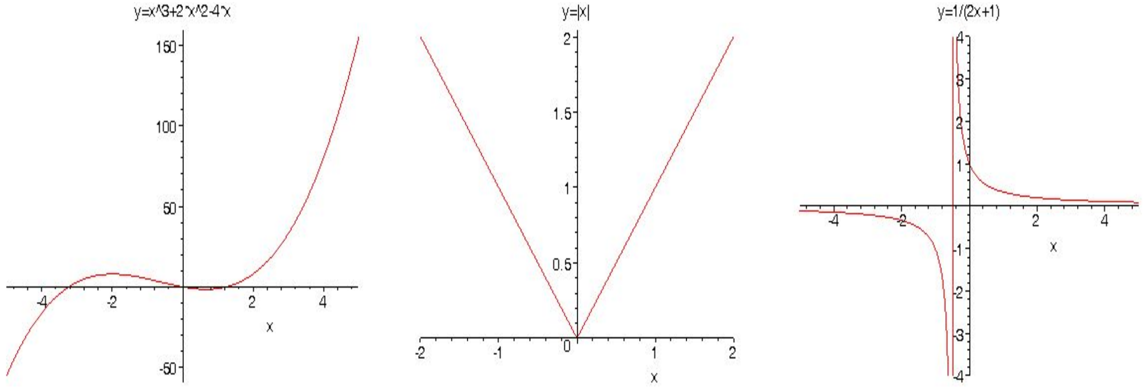

Figure 1: Graphs for question 1

and x^3-1 are negative, and thus the function is positive. Similarly, the function is positive when $x>1$ as both the numerator and denominator are positive. When $x \to 0^+$, $\ln(x) \to -\infty$ and $x^3-1 \to -1$ thus $y \to \infty$. When $x \to 1$, both $\ln(x)$ and x^3-1 approaches 0, so we use *H'Lopital*'s rule,

$$\lim_{x=1} \frac{\ln(x)}{x^3-1} = \lim_{x=1} \frac{d\ln(x)/dx}{d(x^3-1)/dx} = \frac{1/x}{3x^2} = 1/3^1.$$

When $x \to \infty$, using *H'Lopital*'s rule again, we have

$$\lim_{x\to\infty} \frac{\ln(x)}{x^3-1} = \lim_{x\to\infty} \frac{1/x}{3x^2} = 0$$

The range S is $(0,1/3)\bigcup(1/3,\infty)$.

Supplement

Let $f(x) = \frac{\ln(x)}{x^3-1}$. It turns out to be a quite interesting function.

First note that

$$\lim_{x=1} \frac{\ln(x)}{x^3-1} = \lim_{x=1} \frac{d\ln(x)/dx}{d(x^3-1)/dx} = \frac{1/x}{3x^2} = 1/3.$$

Now define another function,

$$g(x) = \begin{cases} f(x) & \text{if } x>0 \text{ and } x \neq 1, \\ 1/3 & \text{if } x=1. \end{cases}$$

Obviously $g(x)$ is continuous. Next we show that it is strictly decreasing on $\mathbf{R}_+$. Below we always assume $x>0$.

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[1] Note that this implies $\lim_{x\to 1^+} \frac{\ln(x)}{x^3-1} = \lim_{x\to 1^-} \frac{\ln(x)}{x^3-1}$.

The first derivative of $g(x)$ is

$$g' = \frac{x^3 - 1 - 3\ln(x)x^3}{x(x^3-1)^2}.$$

Take derivative of the numerator, we can obtain

$$foc = -9\ln(x)x^2.$$

Obviously $foc < 0$ when $x > 1$ and $foc > 0$ when $x < 1$. Moreover, the numerator is zero when $x = 1$. These imply that the numerator is negative whenever $x \neq 1$. The denominator is always nonnegative. Thus g' is always negative when $x \neq 1$.

Note that g' is in a situation of $\frac{0}{0}$ when $x = 1$. Using *H'Lopital*'s rule twice and simplify, we can obtain,

$$\lim_{x=1} g' = \lim_{x=1} -\frac{3x(1+2\ln(x))}{2(7x^3-4)} = -\frac{1}{2} < 0.$$

Thus $g' < 0$ when $x = 1$ as well.

The only difference b/w $g(x)$ and $f(x)$ is the point $(1, 1/3)$, i,e., when $x = 1$, $f(x)$ is not defined, but $g(x)$ is and $g(x) = 1/3$. Since $g(x)$ strictly decreases, no other x can gives $g(x) = 1/3$.

Combine this with the limit of $f(x)$ before when $x \to 0$ and $x \to \infty$. We get the previous results.

Intuitively the function $g(x)$ is CD1 as well.

3. i) $dy/dx = 8x + 3 - 6x^{-1/4} + 3/2x^{-1/2}$;

ii) $dy/dx = 4x^3 + 9x^2 + 4x + 3$;

iii) $dy/dx = -\frac{x^2+1}{(x-1)^2(x+1)^2}$.

4.

$$i) \quad \lim_{x\to 0^-} y = \lim_{x\to 0^-}(-x^2) = 0 = y(0)$$

This indicates that the function is continuous at $x = 0$ and thus everywhere.

$$\lim_{x\to 0^-} \frac{dy}{dx} = \lim_{x\to 0^-}(-2x) = 0 = \lim_{x\to 0^+} 2x = \lim_{x\to 0^+} \frac{dy}{dx}.$$

Then the function is differentiable at $x = 0$ and thus everywhere else.

The derivative is $\frac{dy}{dx} = 2x$ when $x \geq 0$ and $-2x$ when $x < 0$. It can be easily shown that this derivative is continuous at $x = 0$ and everywhere else. So the function is continuously differentiable.

$$\text{ii)}\ \lim_{x\to 0^-} y = \lim_{x\to 0^-}(-x^2-1) = -1 \neq 1 = y(0),$$

Then the function is discontinuous, thus not differentiable, and not continuously differentiable.

iii) The function is continuous, differentiable and continuously differentiable. The steps are similar to those in i) and are skipped.

5. a) Set $y = f(x) = \frac{1}{x+1}$ and solve for x, we can obtain the inverse function as $x = g(y) = \frac{1}{y} - 1$. The domain for the function $y = \frac{1}{x+1}$ is $D = (-\infty, -1)\bigcup(-1, \infty)$ and the range is $S = (-\infty, 0)\bigcup(0, \infty)$. The domain and range for the inverse function is exactly reversed. Calculate the derivatives for both functions:

$$f'(x) = -\frac{1}{(x+1)^2}, \quad g'(y) = -\frac{1}{y^2} = -(x+1)^2 = \frac{1}{f'(x)}.$$

This confirms part (c) of the inverse function theorem on page 79.

b) $y = f(x) = x^{3/4}$

The domain is $D = \mathbf{R}_+$, i.e., all nonnegative real numbers. The range S is also $\mathbf{R}_+$. Solve for x from $y = x^{3/4}$, we obtain $x = g(y) = y^{4/3}$. Its domain and range should also be $\mathbf{R}_+$. Part (c) of the inverse function theorem can be easily verified as the following,

$$f'(x) = \frac{3}{4}x^{-1/4}, \quad g'(y) = \frac{4}{3}y^{1/3} = \frac{4}{3}(x^{3/4})^{1/3} = \frac{4}{3}x^{1/4} = \frac{1}{f'(x)}.$$

Functions in c) and d) are **nonmonotonic** and their inverses do not exist. Of course, you can also choose to specify the domain to be $\mathbf{R}^1_+$, then their inverse functions are as the following:

(c) $g(y) = \frac{-1+\sqrt{4y-7}}{2}$;

(d) $g(y) = \sqrt{log_a y + 2}$.

Verification of the inverse function theorem is similar to above and is skipped.

Extra questions

1. <u>Continuous</u>

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\frac{1}{x} = 0 = f(0).$$

<u>Differentiable</u>

At $x = 0$, left derivative and right derivative are,

$$\lim_{\Delta x\to 0^-} \frac{(\Delta x)^2 \sin\frac{1}{\Delta x} - 0}{\Delta x - 0} = 0,$$

$$\lim_{\Delta x \to 0^+} \frac{(\Delta x)^2 \sin \frac{1}{\Delta x} - 0}{\Delta x - 0} = 0.$$

Since they are equal, the function is differentiable at $x = 0$. It obviously is CD1 thus differentiable everywhere else.

Not CD1

We calculated that the derivative is 0 at $x = 0$. However, the derivative is

$$2x \sin \frac{1}{x} - \cos \frac{1}{x}, \text{ if } x \neq 0,$$

which does not converge as $x \to 0$. Thus the function is not CD1.

2.

Continuous

$$\lim_{x \to 0^-} = 0 = f(0).$$

CD1

$$f'(x) = \begin{cases} x & \text{if } x > 0, \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0. \end{cases}$$

Not CD2

$$f''(x) = \begin{cases} 1 & \text{if } x > 0, \\ \text{does not exist} & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$