Solution to problem Set 2

ECON 5153

1. Their inverses are:

$$A^{-1} = \begin{pmatrix} 3/2 & -5/6 & 1/6 \\ 1/2 & -1/6 & -1/6 \\ -1/4 & 1/4 & 1/4 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} -17 & -3 & 15 \\ -11 & -2 & 10 \\ 10 & 2 & -9 \end{pmatrix}.$$

2. a) First we want to show that $AB^k = B^kA$ by applying $AB = BA$ k times,

$$AB^k = (AB)B^{k-1} = BAB^{k-1} = B(AB)B^{k-2} = B(BA)B^{k-2} = B^2AB^{k-2} = \cdots = B^kA$$

Next we want to prove that if $(AB)^k = A^kB^k$, then $(AB)^{k+1} = A^{k+1}B^{k+1}$.

$$(AB)^{k+1} = (AB)^kAB = A^kB^kAB = A^k\left(B^kA\right)B$$

Using the result above, it equals

$$A^k\left(AB^k\right)B = A^{k+1}B^{k+1}.$$

Note that $(AB)^k = A^kB^k$ holds for $k = 1$, by induction it holds for any $k > 1$.

b) Find arbitrary matrices A, B such that $AB \neq BA$, then compare $(AB)^2$ with A^2B^2.

c)

$$(A+B)^2 = (A+B)(A+B) = AA + AB + BA + BB = A^2 + AB + BA + B^2.$$

It does not equal $A^2 + 2AB + B^2$ unless $AB = BA$.

3. (1) It can be showed that

$$det(A) = -x(-1+x)(x-6).$$

When $x = 0$, $x = 1$ or $x = 6$, the determinant is zero, meaning A is not full rank $r(A) < 3$. We will check $x = 0$ for example. The matrix becomes

$$A = \begin{pmatrix} 5 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

If we just pick the first two rows, we can see that the rank is at least 2. Since $r(A) < 3$, then it must be $r(A) = 2$.

(2) The system of equations can be rewritten as

$$\begin{pmatrix} 1 & 3 & -2 \\ 3 & -17 & 8 \\ 3 & -4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 49 \\ c \end{pmatrix}$$

Let A denote the coefficient matrix, it can be easily showed that $det(A) = 0$. Note that this is a non-homogeneous case ($\mathbf{b} \neq 0$). Thus the system can never have a unique solution.

Since this is a linear system, A is not full rank means that at least one equation is redundant, i.e., it can be recovered by a linear combination of the two equations. We will assume that

$$a \times equation\ 1 + b \times equation\ 3 \Rightarrow equation\ 2, \text{ i.e.,}$$

$$x : a + 3b = 3; \quad y : 3a - 4b = -17; \quad z : -2a + b = 8.$$

Solve for a, b, we can obtain $a = -3, b = 2$. The righthand side for the combination is

$$a \times 9 + b \times c = 49 \Rightarrow -3 \times 9 + 2c = 49 \Rightarrow 2c - 27 = 49.$$

If $2c - 27 \neq 49$, i.e., $c \neq 38$, then the system is inconsistent, and there is no solution.

If $c = 38$, then the equations are consistent, and there are infinitely many solutions. The general solution can be derived as the following: removing one redundant equation (say equation 2). It can be showed that $r(A) = 2$, so one variable can not be solved. We will keep x, and solve y and z as functions of x. Equations 1 and 2 are

$$x + 3y - 2z = 9 \Rightarrow 3y - 2z = 9 - x.$$

$$3x - 4y + z = 38 \Rightarrow -4y + z = 38 - 3x$$

Combine these two equations, we get

$$y = \frac{7x}{5} - 17, z = \frac{13x}{5} - 30.$$

4. (1) a) $\mathbf{u} \cdot \mathbf{v} = (1, 0) \cdot (2, 2) = 1 \times 2 + 0 \times 2 = 2 > 0$, so the angle between them is acute.

b) $\mathbf{u} \cdot \mathbf{v} = (1, -1, 0) \cdot (1, 2, 1) = 1 \times 1 + (-1) \times 2 + 0 \times 1 = -1 < 0$, so the angle between them is obtuse.

(2) First we look for the nonparametric equation. Note that we are dealing with a subspace in $\mathbf{R}^3$. Pick any point on this plane, denote it by $\mathbf{x} = (x_1, x_2, x_3)$. Calculate the displacement

from $\mathbf{p} = (1,3,2)$ to $\mathbf{x}$, which is $(x_1 - 1, x_2 - 3, x_3 - 2)$. View it as a vector $\overrightarrow{\mathbf{px}}$, which is on this plane. Since the vector $\mathbf{n} = (1,-1,0)$ is normal to the plane, it must be perpendicular to the vector $\overrightarrow{\mathbf{px}}$. Therefore,

$$\mathbf{n} \cdot \overrightarrow{\mathbf{px}} = (x_1 - 1) \times 1 + (x_2 - 3) \times (-1) + (x_3 - 2) \times 0 = 0.$$

Simplify it, we obtain the nonparametric equation of the plane $x_1 - x_2 + 2 = 0$.

Next we look for the parametric equation. We need two linearly independent vectors on this plane. There are infinitely many such vectors. First we pick two points on the plane ($x_1 - x_2 + 2 = 0$ must hold), for example,

$$\mathbf{v} = (1,3,0), \quad \mathbf{w} = (2,4,1).$$

Next we generate two vectors connecting either of these two points with point $\mathbf{p}$. Since the plane goes through all these points, the vectors $\overrightarrow{\mathbf{pv}}$ and $\overrightarrow{\mathbf{pw}}$ are both on the plane. The two vectors are

$$\overrightarrow{\mathbf{pv}} = (0,0,-2), \quad \overrightarrow{\mathbf{pw}} = (1,1,-1),$$

and they are linearly independent.

Now we know that the plane goes through point $\mathbf{p}$, and two vectors $\overrightarrow{\mathbf{pv}}$ and $\overrightarrow{\mathbf{pw}}$. The parametric equation for the plane is,

$$\mathbf{x} = (x_1, x_2, x_3) = \mathbf{p} + s\overrightarrow{\mathbf{pv}} + t\overrightarrow{\mathbf{pw}}, \forall s,t \in \mathbf{R^1}.$$

5. a) Suppose that $c_1(2,1) + c_2(1,2) = \mathbf{0}$. Then $(2c_1 + c_2, c_1 + 2c_2) = (0,0)$. This implies $c_1 = c_2 = 0$. Thus the two vectors are linearly independent.

b) Note that $(-4,-2) = 2 \times (2,1)$, the two vectors are linearly dependent.

c) Assume that $c_1(1,1,0) + c_2(0,1,1) = \mathbf{0}$. Then $(c_1, c_1 + c_2, c_2) = (0,0,0) \Rightarrow c_1 = c_2 = 0$. The two vectors are linearly independent.

d) Put the three vectors together into a 3×3 matrix, and calculate its determinant,

$$det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 2 \neq 0.$$

Thus the three vectors are linearly independent. One can also show this by following the method employed in a) and b).