Solution to problem Set 3

ECON 5153

1. In finite dimensional space such as R^n, *closed* and *bounded* characterize compact sets. So it suffices to show that the finite intersection of closed and bounded sets is closed and bounded. The bounded part is obvious. Next we show that the intersection is closed (this is also Theorem 12.10 on page 268). Let $S_1, \cdots, S_n$ be closed sets, and let $S = \bigcap_{i=1}^{n} S_i$. Now suppose that $\{\mathbf{x}_n\}_{n=1}^{\infty}$ is a convergent sequence completely contained in S. Since $S = \bigcap_{i=1}^{n} S_i$, $\{\mathbf{x}_n\}_{n=1}^{\infty}$ must be contained in each S_i, and the limit must be contained in each S_i as well (since each S_i is closed), and thus in S.

Let $S_i = [\frac{i-1}{i}, \frac{i}{i+1}]$, $i = 1, 2, \cdots$. Obviously each S_i is a compact set, but $\bigcup_{i=1}^{\infty} S_i = [0, 1)$ is not compact since it's not closed.

2. (i) The sets A and B are plotted in Figure 1. From the graph, we can easily see that the intersection $A \cap B$ is convex, but their union $A \cup B$ is not convex.

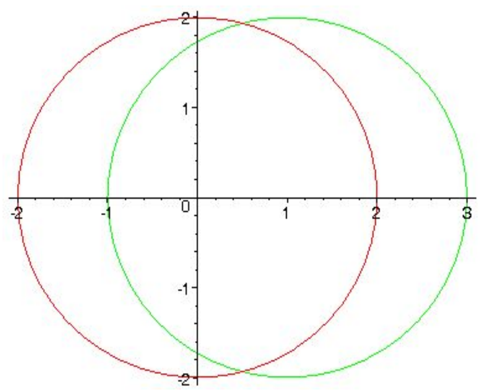

Figure 1: Two circles.

(ii) Set B is plotted in Figure 2.

From the graph, we can easily see that the intersection $A \cap B$ is convex. Their union $A \cup B$ is also convex. This is less obvious, and we explain in detail next. Since both A and B are convex, it suffices to prove that pick any element in A but not in $A \cap B$, and another element in B but not in $A \cap B$, the linear combinations of these two elements must be all in $A \cup B$. Let's first divide the circle into 4 parts, and we only consider the top right, and the bottom left. Linear combinations involving the other two parts are obviously in $A \cup B$ from the graph. Now the top right part and bottom left part are symmetric from the graph. So we only prove that picking anything in A but not in $A \cap B$, and another element from the top right circle, the combination is always in $A \cup B$, and we prove the boundary points only. Namely, pick any element (x_{11}, x_{12})

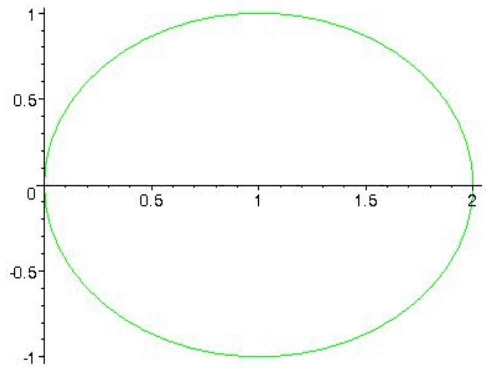

Figure 2: A circle and a square (square is not drawn).

with $x_{11} < 1, x_{12} = 1$, and another element (x_{21}, x_{22}) on the boundary of the top right circle. We need to prove that the linear combinations of (x_{11}, x_{12}) and (x_{21}, x_{22}) are completely in $A \cup B$. Connecting these two points, the line must intersect the circle twice. Since the circle is strictly concave, the line segment is completely below the arc. Thus all the linear combinations are in $A \cup B$.

3. We first calculate $f(x_1, x_2)$ at $(x_1, x_2) = (1, 1)$,

$$f(1,1) = 1^{1/3} \times 1^{2/3} = 1.$$

Now we consider a change from $(1, 1)$ to $(1.1, 0.9)$, i.e. $dx_1 = 0.1$ and $dx_2 = -0.1$. The derivatives evaluated at $(1, 1)$ are,

$$f_1 = 1/3, f_2 = 2/3, f_{11} = -2/9, f_{12} = 2/9, f_{22} = -2/9.$$

Using 2nd order Taylor approximation,

$$\begin{aligned} f(1.1, 0.9) &\approx f(1,1) + (f_1 dx_1 + f_2 dx_2) + \frac{1}{2}[f_{11}(dx_1)^2 + 2 \times f_{12} dx_1 dx_2 + f_{22}(dx_2)^2] \\ &= 1 + 1/3 \times .1 - 2/3 \times .1 + 1/2[-2/9 \times .01 - 4/9 \times .01 - 2/9 \times .01] \\ &= 433/450 \approx 0.9622. \end{aligned}$$

4. A function moves most rapidly when it's moving along the direction of its gradient. First calculate the gradient vector $\nabla F = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)^T = (y^2 + 3x^2 y, 2xy + x^3)^T$. Evaluate it at $(x, y) = (4, -2)$ and simplify, we have $\nabla F = (-92, 48)^T$. The length of the gradient vector is $\sqrt{92^2 + 48^2} = 4\sqrt{673}$. Normalize the vector, we have the direction $\left(\frac{-23}{\sqrt{673}}, \frac{12}{\sqrt{673}}\right)^T \approx (-0.887, 0.463)^T$.

The function would not change if we move along the direction of the level curve, which has the direction $\left(1, -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right)$, or $\left(\frac{\partial f}{\partial y}, -\frac{\partial f}{\partial x}\right)$. Using the numbers above, the direction (after normalizing the length) should be $\left(\frac{12}{\sqrt{673}}, \frac{23}{\sqrt{673}}\right) \approx (0.463, 0.887)$.

5.

a) $6^2 - 3^2 + y^3 = 0 \Rightarrow y^3 = -27$. Pick only the real solution, we have $y = -3$.

b) Solve for y, we have $y = (x_1^2 - x_2^2)^{-1/3}$, this function is well-defined around $x_1 = 6, x_2 = 3$.

c) First calculate all partial derivatives. We can obtain

$$\frac{\partial F}{\partial x_1} = 2x_1, \frac{\partial f}{\partial x_2} = -2x_2 \text{ and } \frac{\partial F}{\partial y} = 3y^2.$$

Then

$$\frac{\partial y}{\partial x_1}(6,3) = -\frac{\partial F/\partial x_1}{\partial F/\partial y} = -\frac{2x_1}{3y^2} = -\frac{2 \cdot 6}{3 \cdot (-3)^2} = -4/9 \text{ and,}$$

$$\frac{\partial y}{\partial x_2}(6,3) = -\frac{\partial F/\partial x_2}{\partial F/\partial y} = -\frac{-2x_2}{3y^2} = -\frac{-2 \cdot 3}{3 \cdot (-3)^2} = 2/9.$$

d) $0 \equiv \frac{\partial F}{\partial x_1}\Delta x_1 + \frac{\partial F}{\partial x_2}\Delta x_2 + \frac{\partial F}{\partial y}\Delta y \Rightarrow 12 \cdot (6.2 - 6) + (-6) \cdot (2.9 - 3) + (3 \cdot (-3)^2) \cdot \Delta y = 0 \Rightarrow \Delta y = -1/9$.

6. Let f_i, $i = 1, 2$ denote the first derivatives of $f(z_1, z_2)$, and f_{ij}, $i, j = 1, 2$ denote the second derivatives. $f(z_1, z_2)$ is strictly concave iff. $f_{11} < 0, f_{22} < 0, f_{11}f_{22} - f_{12}f_{21} > 0$. Take derivatives of the profit function with the two choice variables z_1 and z_2, we have

$$\frac{\partial \pi}{\partial z_1} = 0 \Rightarrow pf_1 - w_1 = 0 \Leftarrow F_1(z_1, z_2, p, w_1, w_2) = 0.$$

$$\frac{\partial \pi}{\partial z_2} = 0 \Rightarrow pf_2 - w_2 = 0 \Leftarrow F_2(z_1, z_2, p, w_1, w_2) = 0.$$

The comparative statics imply that

$$pf_{11}dz_1 + pf_{12}dz_2 + f_1 dp - dw_1 = 0.$$

$$pf_{21}dz_1 + pf_{22}dz_2 + f_2 dp - dw_2 = 0.$$

Arrange them in matrix form, we can obtain

$$\begin{pmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{pmatrix} \begin{pmatrix} dz_1 \\ dz_2 \end{pmatrix} = \begin{pmatrix} -f_1 & 1 & 0 \\ -f_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} dp \\ dw_1 \\ dw_2 \end{pmatrix}.$$

$$\Rightarrow \begin{pmatrix} dz_1 \\ dz_2 \end{pmatrix} = \begin{pmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{pmatrix}^{-1} \begin{pmatrix} -f_1 & 1 & 0 \\ -f_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} dp \\ dw_1 \\ dw_2 \end{pmatrix}.$$

Use adjoint matrix to solve for the inverse, we have

$$\begin{pmatrix} dz_1 \\ dz_2 \end{pmatrix} = \frac{1}{p(f_{11}f_{22} - f_{12}f_{21})} \begin{pmatrix} f_{22} & -f_{12} \\ -f_{12} & f_{11} \end{pmatrix} \begin{pmatrix} -f_1 & 1 & 0 \\ -f_2 & 0 & 1 \end{pmatrix} \begin{pmatrix} dp \\ dw_1 \\ dw_2 \end{pmatrix}.$$

$$= \frac{1}{p(f_{11}f_{22} - f_{12}f_{21})} \begin{pmatrix} -f_1 f_{22} + f_2 f_{12} & f_{22} & -f_{12} \\ f_1 f_{12} - f_2 f_{11} & -f_{12} & f_{11} \end{pmatrix} \begin{pmatrix} dp \\ dw_1 \\ dw_2 \end{pmatrix}.$$

From above, we can get all 6 partial derivatives, $\frac{\partial z_i}{\partial p}$, $\frac{\partial z_i}{\partial w_1}$ and $\frac{\partial z_i}{\partial w_2}$, $i = 1, 2$. Specifically

$$\frac{\partial z_1}{\partial w_2} = \frac{\partial z_2}{\partial w_1} = \frac{-f_{12}}{p(f_{11}f_{22} - f_{12}f_{21})}.$$

(2) Note that $f_{11}f_{22} - f_{12}f_{21} > 0$, the sign of the partial derivatives only depend on f_{12}, and they are both positive when $f_{12} < 0$. This means that the two inputs are substitutes and that is why when the price of the .rst input increases the demand of the second input increases.