Solution to problem Set 4

ECON 5153

1. a) $|A_1| = 1 > 0$, $|A_2| = |A| = -14 < 0$. This implies that the matrix is indefinite.

b) $|A_1| = 1$, $|A_2| = 0$. The patterns of p.d. or n.d. are violated by a zero entry. Therefore, we know that it's not p.d. or n.d.[1] Next we check for semidefiniteness, and we need to check all principal submatrices, not just leading principal submatrices.

1) There are three $|A_1|$'s (1,1 and 2), which are all positive.

2) There are three $|A_2|$'s, by deleting i_{th} ($i = 1, 2, 3$) row and column:

$$det \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = 2, \quad det \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 0, \quad \text{and } det \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = 2.$$

3) There is only one $|A_3| = 0$.

Since $|A_i| \geq 0$ for all $i = 1, 2, 3$, we know that the matrix is p.s.d. It is thus not indefinite. It is not n.s.d. since some principal minors are positive.[2]

2. a) Check the determinant of the bordered matrix

$$det \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & -1 \\ 2 & -1 & -4 \end{pmatrix} = -4 < 0,$$

thus the quadratic form is p.d. on the constraint.

b) Note that there are $n = 3$ variables, and $m = 2$ constraints, and we need to check the last $n - m = 1$ leading principal matrix of the bordered matrix, which is the $(3 + 2) \times (3 + 2)$ bordered matrix itself. It can be shown that

$$det \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & -2 & 0 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 \end{pmatrix} = 0.$$

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[1] This does not necessarily mean that the matrix is indefinite.

[2] Note that the quadratic form correspondent to the matrix is $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 = (x_1 - x_2)^2 + x_3$, which is clearly nonnegative and thus p.s.d.

The quadratic form is both p.s.d and n.s.d. on the constraints. It can be verified that after substituting the constraints, $Q = 0$ always holds.

3. a) $f(x,y) = x^4 + x^2 - 6xy + 3y^2$.

$f_x = 4x^3 + 2x - 6y$ and $f_y = -6x + 6y$. Set $f_y = 0$, we have $y = x$. Then $f_x = 0$ implies $4x^3 - 4x = 0$ or $x = \pm 1, 0$.

Three critical points: (-1,-1), (0,0) and (1,1).

$$Hessian : H = \begin{pmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{pmatrix}.$$

$$H(-1,-1) = H(1,1) = \begin{pmatrix} 14 & -6 \\ -6 & 6 \end{pmatrix}, \text{ H is p.d., thus (-1,-1) and (1,1) are local min.}$$

$$|H(0,0)| = det \begin{pmatrix} 2 & -6 \\ -6 & 6 \end{pmatrix} < 0, \text{ implying H is indefinite, so (0,0) is a saddle point.}$$

(b) $f(x,y) = x^2 - 6xy + 2y^2 + 10x + 2y - 5$.

$f_x = 2x - 6y + 10$ and $f_y = -6x + 4y + 2$.

Set $f_x = f_y = 0$, and solve for x, y, we can obtain $(x,y) = (13/7, 16/7)$. This is the only critical point.

$$Hessian : H = \begin{pmatrix} 2 & -6 \\ -6 & 4 \end{pmatrix}, \text{ which is indefinite, so the point is a saddle point.}$$

(c) $f(x,y) = xy^2 + x^3y - xy$.

$f_x = y(3x^2 + y - 1)$ and $f_y = x(x^2 + 2y - 1)$.

Set $f_x = f_y = 0$. There are four cases:

Case 1: $x = y = 0$.

Case 2: $x = 0$ and $3x^2 + y - 1 = 0$, i.e., $(x,y) = (0,1)$.

Case 3: $y = 0$ and $x^2 + 2y - 1 = 0$, i.e., $(x,y) = (1,0)$ or $(-1,0)$.

Case 4: $3x^2 + y - 1 = 0$ and $x^2 + 2y - 1 = 0$. This implies that $(x, y) = (1/\sqrt{5}, 2/5)$ or $(-1/\sqrt{5}, 2/5)$.

There are a total of six critical points: (0,0), (0,1), (1,0), (-1,0), $(-1/\sqrt{5}, 2/5)$ and $(1/\sqrt{5}, 2/5)$.

$$Hessian : H = \begin{pmatrix} 6xy & 3x^2 + 2y - 1 \\ 3x^2 + 2y - 1 & 2x \end{pmatrix}.$$

$$\text{At } (0,0), H = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \text{ which is indefinite, so (0,0) is a saddle.}$$

$$\text{At } (0,1), H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ which is indefinite, so (0,1) is a saddle.}$$

$$\text{At } (1,0), H = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \text{ which is indefinite, so (1,0) is a saddle.}$$

$$\text{At } (-1,0), H = \begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix}, \text{ which is indefinite, so (-1,0) is a saddle.}$$

$$\text{At } (1/\sqrt{5}, 2/5), H = \begin{pmatrix} \frac{12}{5\sqrt{5}} & 2/5 \\ 2/5 & 2/\sqrt{5} \end{pmatrix}, \text{ which is p.d., so } (1/\sqrt{5}, 2/5) \text{ is a local min.}$$

$$\text{At } (-1/\sqrt{5}, 2/5), H = \begin{pmatrix} -\frac{12}{5\sqrt{5}} & 2/5 \\ 2/5 & -2/\sqrt{5} \end{pmatrix}, \text{ which is n.d., so } (-1/\sqrt{5}, 2/5) \text{ is a local max.}$$

(d) $f(x, y) = 3x^4 + 3x^2y - y^3$.

$f_x = 12x^3 + 6xy$ and $f_y = 3x^2 - 3y^2$.

Set $f_y = 0$, we have $x = \pm y$.

Case 1: $x = y$. $f_x = 0$ implies $x = 0$ or $-1/2$.

Case 2: $x = -y$. $f_x = 0$ implies that $x = 0$ or $1/2$.

There are three critical points: (0,0), (-1/2,-1/2) and (1/2,-1/2).

$$Hessian : H = \begin{pmatrix} 36x^2 + 6y & 6x \\ 6x & -6y \end{pmatrix}.$$

$$\text{At } (0,0), H = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

which is indeterminate. However, $f(0, y) = -y^3$, so (0,0) is neither a max nor a min.

$$\text{At } (-1/2, -1/2), H = \begin{pmatrix} 6 & -3 \\ -3 & 3 \end{pmatrix}, \text{ which is p.d., so } (-1/2, -1/2) \text{ is a local min.}$$

$$\text{At } (1/2, -1/2), H = \begin{pmatrix} 6 & 3 \\ 3 & 3 \end{pmatrix}, \text{ which is p.d., so } (1/2, -1/2) \text{ is a local min.}$$

4. Solution to the question without assigning values of prices

Firm's problem is the following:

$$\begin{aligned} \max_{L,K} \pi &= pf(L, K) - (wL + rK) \\ &= p \cdot 4L^{1/4}K^{1/4} - wL - rK \end{aligned}$$

Note that $\lim_{L \to 0^+} \frac{\partial \pi}{\partial L} \to +\infty$ and $\lim_{K \to 0^+} \frac{\partial \pi}{\partial K} \to +\infty$. This implies that the solution will be interior. It can be shown that the Hessian is n.d., which means that F.O.C. is necessary and sufficient and leads to a global maximum.

First order conditions imply

$$\frac{\partial \pi}{\partial L} = 0 \Rightarrow pL^{-3/4}K^{1/4} = w. \tag{1}$$

$$\frac{\partial \pi}{\partial K} = 0 \Rightarrow pL^{1/4}K^{-3/4} = r. \tag{2}$$

Equation (1) divided by (2) implies,

$$\frac{K}{L} = \frac{w}{r} \Rightarrow K = \frac{w}{r}L.$$

Substitute this into equation (1), we can obtain

$$pL^{-3/4}\left(\frac{w}{r}\right)^{1/4}L^{1/4}=w\Rightarrow p\left(\frac{w}{r}\right)^{1/4}L^{-1/2}=w$$

$$\Rightarrow \boxed{L=p^2w^{-3/2}r^{-1/2}}.$$

Then

$$K=\frac{w}{r}L\Rightarrow \boxed{K=p^2w^{-1/2}r^{-3/2}}.$$

The resulting output is

$$f(L,K)=4L^{1/4}K^{1/4}\Rightarrow \boxed{Q=4pw^{-1/2}r^{-1/2}},$$

and the profit is

$$\pi=pQ-wL-rK=p4pw^{-1/2}r^{-1/2}-wp^2w^{-3/2}r^{-1/2}-rp^2w^{-1/2}r^{-3/2}$$

$$\Rightarrow \boxed{\pi=2p^2w^{-1/2}r^{-1/2}}.$$

<u>An example with specific prices</u>

Production function is $f(L,K)=3L^{1/3}K^{1/3}$. Output price is $p=1$, input prices are $w=r=2$.

Firm's problem is the following:

$$\max_{L,K}\pi=pf(L,K)-(wL+rK)$$

$$\text{subject to } f(L,K)=3L^{1/3}K^{1/3}$$

$$L\geq 0, K\geq 0.$$

One can use the Lagrangian, but an easier way to solve this problem is to directly plug $y=L^{1/3}K^{1/3}$ into the profit function and use the prices. Then firm's problem becomes the following,

$$\max_{L,K}\pi=3L^{1/3}K^{1/3}-2L-2K$$

$$\text{subject to } L\geq 0, K\geq 0.$$

Note that $\lim_{L\to 0^+} \frac{\partial \pi}{\partial L} \to +\infty$ and $\lim_{K\to 0^+} \frac{\partial \pi}{\partial K} \to +\infty$. This implies that the solution will be interior. It can be shown that the Hessian is n.d., which means that F.O.C. is necessary and sufficient and leads to a global maximum.

Solve the first order conditions

$$\frac{\partial \pi}{\partial L} = \frac{\partial \pi}{\partial K} = 0,$$

we can obtain $L^* = K^* = 1/8$ and $\pi = 1/4$.