**Solution to problem set 5**

ECON 5153

**1**. It would be easier to substitute $x = 1/2$ and transform the question into

$$\max \quad f(y,z) = \frac{yz}{2}$$

$$\text{subject to } y + z = 1/2.$$

One can show that the optimal choice is $y^* = z^* = 1/4$.

Alternatively, one can keep all the constraints, and form the Lagrangian $L = xyz - \mu_1(x + y + z - 1) - \mu_2(x - 1/2)$.

First order conditions are:

$$\frac{\partial L}{\partial x} = yz - \mu_1 - \mu_2 = 0 \tag{1}$$

$$\frac{\partial L}{\partial y} = yz - \mu_1 = 0 \tag{2}$$

$$\frac{\partial L}{\partial z} = yz - \mu_1 = 0 \tag{3}$$

$$\frac{\partial L}{\partial \mu_1} = x + y + z - 1 = 0 \tag{4}$$

$$\frac{\partial L}{\partial \mu_2} = x - 1/2 = 0 \tag{5}$$

(2) $\Rightarrow x^* = 1/2$. Equations (2) and (3) imply $y = z$. Plug $x^* = 1/2$ and $y = z$ into (4), we get $y^* = z^* = 1/4$. Then $f(x^*, y^*, z^*) = x^*y^*z^* = 1/32$.

**2**.

$$\max \quad f(x,y) = -2x^2 + y$$

$$\text{subject to } 3x - y = 1, x \geq 0, y \geq 0.$$

First form the Lagrangian $L = -2x^2 + y - \mu(3x - y - 1) + \lambda_1 x + \lambda_2 y$.

There are several cases, depending on whether the inequality constraints are binding or not. Clearly in all cases, NDCQ is satisfied.

Case 1: $x \geq 0$ is binding, then $x = 0$. Then $3x - y = 1 \Rightarrow y = -1$. But this violates the constraint $y \geq 0$.

Case 2: $y \geq 0$ is binding, then $y = 0$. Then $x = 1/3 \Rightarrow f(x,y) = -2/9$.

Case 3: None inequality constraint is binding ($x > 0, y > 0$). This implies that $\lambda_1 = \lambda_2 = 0$. Now we take first order conditions:

$$\frac{\partial L}{\partial x} = -4x - 3\mu = 0 \tag{6}$$

$$\frac{\partial L}{\partial y} = 1 + \mu = 0 \tag{7}$$

$$\frac{\partial L}{\partial \mu} = 3x - y = 1 \tag{8}$$

Combine equations (6), (7) and (8), we have $x^* = 3/4$, $y^* = 5/4$ (note that $x > 0$ and $y > 0$ are satisfied). Then $f(x^*, y^*) = 1/8$, which is greater than $-2/9$ from case 2. Next we check the second order condition at $(x^*, y^*)$.

The bordered Hessian is

$$H = \begin{pmatrix} 0 & 3 & -1 \\ 3 & -4 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

The last leading principal minor is

$$det(H) = 4 > 0.$$

This has the same sign as $(-1)^n$, implying that the bordered Hessian is negative definite and $(x^*, y^*)$ is a local maximum. Therefore, $f(x^*, y^*) = 1/8$ is maximum which occurs at $x^* = 3/4$, $y^* = 5/4$.

**3.**

$$\min \quad f(x, y) = -x^2 - y^2$$

$$\text{subject to } 2x + y \leq 2, x \geq 0, y \geq 0.$$

First form the Lagrangian[1]

$$L = -x^2 - y^2 + \lambda_1(2x + y - 2) - \lambda_2(x - 0) - \lambda_3(y - 0).$$

There are several cases:

Case 1: $x \geq 0$ is binding, i.e., $x = 0$. Then $2x + y \leq 2 \Rightarrow y \in [0, 2] \Rightarrow f(x, y) \geq -4$.

Case 2: $y \geq 0$ is binding, that is, $y = 0$. Then $2x + y \leq 2 \Rightarrow x \in [0, 1] \Rightarrow f(x, y) \geq -1$.

[1]Note that this is a minimization problem and the way we form Lagrangian is a little different from that in maximization problems.

Case 3: $x \geq 0$ and $y \geq 0$ are not binding, but $2x + y \leq 2$ is binding. This implies that $\lambda_2 = \lambda_3 = 0$, but $\lambda_1 \geq 0$ and $\frac{\partial L}{\partial \lambda_1} = 0$.

Take the first order conditions:

$$\frac{\partial L}{\partial x} = -2x + 2\lambda_1 = 0 \tag{9}$$

$$\frac{\partial L}{\partial y} = -2y + \lambda_1 = 0 \tag{10}$$

$$\frac{\partial L}{\partial \lambda_1} = 2x + y - 2 = 0 \tag{11}$$

Combine equations (9), (10) and (11), we have $x^* = 4/5$, $y^* = 2/5$ and $\lambda_1^* = 4/5$ (note that $x > 0$ and $y > 0$ are satisfied). Then $f(x^*, y^*) = -4/5$.

Case 4: $x \geq 0$ and $y \geq 0$ are not binding, $2x + y \leq 2$ is not binding either.

This case can't be optimal, as $\forall x > 0, y > 0$ and $2x + y < 2$, it can be shown that there exists $(x, y')$ such that $2x + y' = 2$ with $f(x, y') < f(x, y)$.

Comparing all cases, case 1 provides the minimum, with $(x^*, y^*) = (0, 2)$. At this point, there are two variables $(x, y)$, and two constraints ($n = m = 2$). We cannot perform the second order conditions test, which requires us to check the last $n - m$ leading principal minors. Therefore, $x^* = 0$, $y^* = 2$ and $f(x^*, y^*) = -4$.

**4.**

$$\max \quad f(x_1, x_2) = 2x_1^{1/2} x_2^{1/2}$$

$$\text{subject to } 2x_1 + 3x_2 \leq 4, x_1 \geq 0, x_2 \geq 0.$$

We start by forming the Kuhn-Tucker Lagrangian as the following

$$\tilde{L} = 2x_1^{1/2} x_2^{1/2} - \lambda(2x_1 + 3x_2 - 4).$$

The K-T conditions are

$\frac{\partial \tilde{L}}{\partial x_i} \leq 0$, $x_i \frac{\partial \tilde{L}}{\partial x_i} = 0$, $i = 1, 2$ and $\frac{\partial \tilde{L}}{\partial \lambda} \geq 0$, $\lambda \frac{\partial \tilde{L}}{\partial \lambda} = 0$.

Depending on whether $x_i = 0$, $i = 1, 2$ and whether $2x_1 + 3x_2 \leq 1$ is binding, we have several cases:

Case 1& 2: $x_1 = 0$ or $x_2 = 0$. Then the utility is automatically zero.

Case 3: $x_1 > 0$, $x_2 > 0$, and $2x_1 + 3x_2 \leq 4$ is binding. This implies $\frac{\partial \tilde{L}}{\partial \lambda} = 0$.

The first order conditions are:

$$\frac{\partial \tilde{L}}{\partial x_1} = x_1^{-1/2} x_2^{1/2} - 2\lambda = 0 \tag{12}$$

$$\frac{\partial \tilde{L}}{\partial x_2} = x_1^{1/2} x_2^{-1/2} - 3\lambda = 0 \tag{13}$$

$$\frac{\partial \tilde{L}}{\partial \lambda} = 2x_1 + 3x_2 = 4 \tag{14}$$

Combine equations (12), (13) and (14), we have $x_1^* = 1$, $x_2^* = 2/3$ and $\lambda^* = 6/\sqrt{6}$ (note that $x_1 > 0$ and $x_2 > 0$ are satisfied). Then $f(x_1^*, x_2^*) = 2\sqrt{6}/3$.

Case 4: $x_1 > 0$, $x_2 > 0$ and $2x_1 + 3x_2 \leq 4$ is not binding.

This case can't be optimal, as $\forall x_1 > 0, x_2 > 0$ and $2x_1 + 3x_2 < 4$, it can be shown that there exists $(x_1, x_2')$ such that $2x_1 + 3x_2' = 4$ with $f(x_1, x_2') > f(x_1, x_2)$.

Comparing all cases, case 3 provides the maximum. Therefore, $x_1^* = 1$, $x_2^* = 2/3$ and $f(x^*, y^*) = 2\sqrt{6}/3$.

**5**. 1) First solve the problem when one of the constraints is $x^2 + y^2 + z^2 = 1$. The Lagrangian is

$$L = x + y + z^2 - \mu_1[x^2 + y^2 + z^2 - 1] - \mu_2(y - 0)$$

The first order conditions are:

$$\frac{\partial L}{\partial x} = 1 - 2\mu_1 x = 0,$$

$$\frac{\partial L}{\partial y} = 1 - 2\mu_1 y - \mu_2 = 0,$$

$$\frac{\partial L}{\partial z} = 2z - 2\mu_1 z = 0,$$

$$\frac{\partial L}{\partial \mu_1} = 0 \Rightarrow x^2 + y^2 + z^2 = 1, \text{ and}$$

$$\frac{\partial L}{\partial \mu_2} = 0 \Rightarrow y = 0.$$

Combine all these equations, we can obtain $x = 1/2$, $y = 0$, $z = \pm\sqrt{3}/2$, $\mu_1 = 1$ and $\mu_2 = 1$.[2] By theorem 19.1,

$$f(x, y, z)|_{a=0.8} = f(x, y, z)|_{a=1} + \mu_1 \cdot (0.8 - 1) = (\frac{1}{2} + 0 + \frac{3}{4}) - 0.2 \cdot 1 = 1.05.$$

2) Look at the problem

$$\max \quad x^2 + x + ay^2$$

[2]There are other solutions involving $z = 0$ which can be showed that are not local maximum and we skip them.

$$\text{subject to} \quad 2x + 2y \leq 1, x \geq 0, y \geq 0.$$

The Lagrangian is

$$L = x^2 + x + ay^2 - \lambda_1(2x + 2y - 1) + \lambda_2 x + \lambda_3 y.$$

For $a = 4$ (Example 18.13), the solutions are $x^* = 0$, $y^* = 0.5$, $\lambda_1^* = 2$, $\lambda_2^* = 3$, $\lambda_3^* = 0$ and $f^* = 1$. At these values,

$$\frac{\partial L}{\partial a} = y^2 = 0.25.$$

Then

$$f^*(4.1) \approx f^*(4) + \frac{\partial L}{\partial a} \cdot \Delta a = 1 + 0.25 \cdot 0.1 = 1.025.$$